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A rigidity property of superpositions involving determinantal processes
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The paper under review is about rigidity of point processes. There are many different notions of 'rigidity', e.g., the one in the Ghosh-Peres sense (see [2]) and the notion used in [1] in terms of the order of fluctuations. The 'rigidity' concerned in this paper is in the following sense: suppose we are given independent superposition (i.e., sum) of two point processes, each coming from some prescribed classes, can we recover each of the two processes? The main result of the paper is the following theorem.

Theorem 1. *If (N, Π) is a pair of independent point processes on the same probability space with N Poisson and Π determinantal induced by a locally trace class operator, then the independent superposition $N + \Pi$ uniquely determines N and Π.*

We first explain what it means by 'Poisson' and 'determinantal'. Point process N is called Poisson process with intensity measure ν if for any compact measurable set B, $N(B)$ has Poisson distribution with mean $\nu(B)$ and for any finitely many disjoint sets $B_1, \ldots, B_k$ the random variables $N(B_1), \ldots, N(B_k)$ are independent. We say Π is a determinantal point process on (E, μ) associated with a localy trace class operator $K : L^2(E, \mu) \to L^2(E, \mu)$ if the probability generating functional $\mathcal{B}_\Pi$ of Π satisfies the following

$$\mathcal{B}_\Pi(\phi) := \mathbb{E}\left(\prod_{x\in E}(1+\phi(x))^{\Pi(\{x\})}\right) = \det\left(1+\phi K 1_{\{supp_e(\phi)\}}\right), \forall\ \phi \in \mathbb{B}_c(E).$$

Here $\Pi(\{x\}) = 1$ if $x \in \Pi$ and 0 otherwise, $supp_e(\phi)$ stands for the essential support of the bounded measurable compactly-supported function ϕ and det means the Fredholm determinant. Note there is a more common definition which invovles specification of correlation functions (see [3] for a detailed introduction to determinantal point process).

The proof of the main theorem is actually quite short. Indeed, knowing the distribution of N and Π is equivalent to knowing the value of $\mathcal{B}_N(\phi)$ and $\mathcal{B}_\Pi(\phi)$ for all ϕ. Since N and Π are independent, we have

$$\mathcal{B}_{N+\Pi}(\phi) = \mathcal{B}_N(\phi)\mathcal{B}_\Pi(\phi).$$

Replacing ϕ by $z\phi$ where $z \in \mathbb{C}$, the complex plane, we get

$$\mathcal{B}_{N+\Pi}(z\phi) = \mathcal{B}_N(z\phi)\mathcal{B}_\Pi(z\phi). \tag{1}$$

If we fix ϕ and view z as the variable then we get an equality for the entire functions $\mathcal{B}_{N+\Pi}(z\phi), \mathcal{B}_N(z\phi)$ and $\mathcal{B}_\Pi(z\phi)$ on $\mathbb{C}$.

One can compute $\mathcal{B}_N(z\phi) = \exp\left(z\int_E \phi d\nu\right)$. As a consequence, $\mathcal{B}_N(z\phi) > 0, \forall\ z, \phi$.

On the other side, one can factorize the Fredholm determinant:

$$\mathcal{B}_\Pi(z\phi) = \det\left(1+z\phi K 1_{\{supp_e(\phi)\}}\right) = \prod_{i=1}^{m}(1+zz_i),$$

where $z_i, 1 \leq i \leq m$ are the non-zero eigenvalues (counting multiplicities) of the trace class operator $\phi K 1_{\{supp_e(\phi)\}}$. Thus the function $\mathcal{B}_\Pi(z\phi)$ is uniquely determined by its zeros.

As a consequence, given $\mathcal{B}_{N+\Pi}(z\phi)$ we can recover $\mathcal{B}_\Pi(z\phi)$ since the zeros of the function $\mathcal{B}_\Pi(z\phi)$ and $\mathcal{B}_{N+\Pi}(z\phi)$ are the same (by equation (1) and the fact that $\mathcal{B}_N(z\phi) > 0$). Then we can recover N since

$$\mathcal{B}_N(\phi) = \mathcal{B}_{N+\Pi}(\phi)/\mathcal{B}_\Pi(\phi).$$

One interesting question posed by the author is, whether there is certain physical interpretation for Theorem 1. The determinantal point process is initially used to model fermion systems while Poisson distribution can be thought of as placing n particles independently in a scaled space and let $n \to \infty$. On the mathematical side, it would be helpful to use such interpretation to predict new results.

References

[1] Chatterjee, Sourav. "Rigidity of the three-dimensional hierarchical Coulomb gas." Probability Theory and Related Fields 175.3-4 (2019): 1123-1176.

[2] Ghosh, Subhroshekhar, and Yuval Peres. "Rigidity and tolerance in point processes: Gaussian zeros and Ginibre eigenvalues." Duke Mathematical Journal 166.10 (2017): 1789-1858.

[3] Soshnikov, Alexander. "Determinantal random point fields." Russian Mathematical Surveys 55.5 (2000): 923-975.