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Pseudohyperbolic metric and uniformly discrete sequences in the real unit ball
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This paper considers a Carleson type inequality for functions in harmonic Bergman spaces, i.e. spaces of of all L^p $(p > 0)$ integrable harmonic functions on the unit ball $\mathbb{B}_n$ in $\mathbb{R}^n$, which involves a sequence of points in $\mathbb{B}_n$. A similar problem was considered by Duren and Weir [2] for L^p integrable holomorphic functions on the unit ball in $\mathbb{C}^n$. The authors of this paper are able to establish analogous results for harmonic functions in $\mathbb{R}^n$. To prove their main result, Ren and Kähler established some new properties on automorphisms of the unit ball and the induced pseudohyperbolic metric.

Automorphisms of the unit ball in $\mathbb{C}^n$ have been extensively studied and are an integral part of holomorphic function theory on the unit ball (see [3]). Their real counterparts were given by Ahlfors [1], but have not been well studied and widely known. Much of the paper is devoted to establish properties similar to those for the complex variable case. For x, $a \in \mathbb{B}_n$, let

$$\varphi_\alpha(x) = \frac{|x-a|^2 a - (1-|a|^2)(x-a)}{||x|a - x'|^2}$$

denote an automorphism or Möbius transformation of $\mathbb{B}_n$, where $x = |x|x'$, and let

$$\rho(x,a) = |\varphi_a(x)|$$

denote the pseudohyperbolic metric on $\mathbb{B}_n$. One of the interesting properties proved in the paper is that the pseudohyperbolic metric is invariant under Möbius transformations.

A sequence of points $\{x_k\}$ in $\mathbb{B}_n$ is said to be uniformly discrete if there exists a constant $\delta \in (0,1)$ such that

$$|\varphi_{x_j}(x_k)| \geq \delta$$

for all $j \neq k$. One of the main results of the paper can be stated as follows. There is a constant $C > 0$ such that

$$\sum_{k=1}^{\infty}(1-|x_k|^2)^n|f(x_k)|^p \leq C\|f\|_p^p$$

for all harmonic functions $f \in L^p(\mathbb{B}_n)$ if and only if $\{x_k\}$ is a finite union of uniformly discrete sequences.

The study of automorphisms of the unit ball in $\mathbb{R}^n$ can help us better understand the theory of harmonic functions and operators defined on harmonic functions on the unit ball as in the complex variable case.

References

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[3] Rudin W *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer–Verlag, New York, 1980.