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Hearts of twin cotorsion pairs on exact categories
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In algebra, geometry and topology, exact categories and triangulated categories are two fundamental structures. As expected, exact categories and triangulated categories are not independent of each other. A well known fact is that triangulated categories which are at the same time abelian must be semisimple [Mi]. Also, there are a series of ways to produce triangulated categories from abelian ones, such as, taking the stable categories of Frobenius exact categories [Hap], or taking the homotopy categories or derived categories of complexes over abelian categories [Mi].

On the other hand, because of the recent development of the cluster tilting theory, it becomes possible to produce abelian categories from triangulated ones. For example, in the paper [KR] of Keller and Reiten, it was shown that the quotient of a triangulated category (with some conditions) by a cluster tilting subcategory becomes an abelian category. After that, Koenig and Zhu [KZ] showed in detail, how the abelian structure is given on this quotient category, in a more abstract setting. On the other hand, as is well known since 1980s, the heart of any t-structure is abelian ([BBD]). In [Na11], Nakaoka unified these two constructions by using the notion of a cotorsion pair in triangulated categories.

To any cotorsion pair in a triangulated category, one can naturally associate an abelian category, which gives back each of the above two abelian categories, when the cotorsion pair comes from a cluster tilting subcategory, or a t-structure, respectively.

In fact, the notion of cotorsion pairs first appeared in [Sa], and played an important role to solve the well known Flat Cover Conjecture ([EJ]). Now it has been deeply studied in many aspects of homological algebra and representation theory.

In the paper of under review, the author mainly studied the structure of hearts of a (twin) cotorsion pair on an exact category $\mathcal{B}$. To address it, the author defined two subcategories of $\mathcal{B}$ for a given cotorsion pair $(\mathcal{U}, \mathcal{V})$ as follows:

By the definition of cotorsion pairs, any $B \in \mathcal{B}$ admits two short exact sequences $V_B \rightarrowtail U_B \twoheadrightarrow B$ and $B \rightarrowtail V^B \twoheadrightarrow U^B$ with $U_B, U^B \in \mathcal{U}$ and $V_B, V^B \in \mathcal{V}$. Define

$$\mathcal{B}^+ = \{B \in \mathcal{B} \mid U_B \in \mathcal{V}\}, \mathcal{B}^- = \{B \in \mathcal{B} \mid V^B \in \mathcal{U}\}.$$

Then the author defined the heart of $(\mathcal{U}, \mathcal{V})$ as the quotient category

$$\underline{\mathcal{H}} := \frac{\mathcal{B}^+ \cap \mathcal{B}^-}{\mathcal{U} \cap \mathcal{V}}.$$

The author stated the first main result as follows, which is an analogue of [Na11, Theorem 6.4].

Theorem 1. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair on an exact category $\mathcal{B}$ with enough projectives and injectives. Then $\underline{\mathcal{H}}$ is abelian.*

Moreover, the author considered a pair of cotorsion pairs $(\mathcal{S}, \mathcal{T})$, $(\mathcal{U}, \mathcal{V})$ in $\mathcal{B}$ such that $\mathcal{S} \subseteq \mathcal{U}$, which is called a twin cotorsion pair. The notion of hearts is generalized to such pairs, and the author obtained the second main result as follows, which is an analogue of [Na13, Theorem 5.4].

Theorem 2. *Let $(\mathcal{S}, \mathcal{T})$, $(\mathcal{U}, \mathcal{V})$ be a twin cotorsion pair on $\mathcal{B}$. Then $\underline{\mathcal{H}}$ is semi-abelian.*

The author also considered special cases where the heart has nicer structure, for example, the heart of a special twin cotorsion pair $(\mathcal{S}, \mathcal{T})$, $(\mathcal{U}, \mathcal{V})$ is integral and almost abelian. Finally the author showed that the Gabriel-Zisman localization of the heart at the class of regular morphisms is abelian, and moreover it is equivalent to the category of finitely presented modules over a stable subcategory of $\mathcal{B}$. The author also gave some examples of twin cotorsion pairs in the last section.

The results in this paper are very interesting and it is subtle in the aspect of dealing with certain details.

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