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Poisson stable motions of monotone nonautonomous dynamical systems
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Let $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and consider the following equation

$$x' = f(t, x). \tag{1}$$

The existence of Bohr almost periodic solutions of the equation (1) with Bohr almost periodic right-hand side f in t, uniformly with respect to x on every compact subset of $\mathbb{R}^n$, was studied by many authors (see e.g. [4, 7, 8, 9]).

Opial [4] considered the scalar (i.e., $n = 1$) case and showed that if f is monotone with respect to the spacial variable x then every bounded solution (on the whole real axis) is Bohr almost periodic.

Recall that a function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is called *regular* if for any $g \in H(f)$ and $v \in \mathbb{R}^n$ the limit equation

$$x' = g(t, x) \tag{2}$$

admits a unique solution $\varphi(t, v, g)$ defined on $\mathbb{R}$ with initial value $x(0) = v$, where $H(f) = \overline{\{f^\tau : \tau \in \mathbb{R}\}}$, $f^\tau(t, x) = f(t + \tau, x)$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, and by bar we mean the closure in $C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ endowed with the compact-open topology.

Zhikov [9] studied the scalar equation (1) with regular f but without the monotonicity assumption for f. He obtained the existence of at least one almost periodic solution of (1) if it admits one bounded and uniformly stable solution.

Sacker and Sell [5] generalized Zhikov's result, still for scalar equations, by replacing the regularity of f with positive regularity. In addition, Sacker and Sell [6] studied almost periodic solutions for the general case (not necessarily scalar) of (1) in the framework of skew-product flows.

Shcherbakov [7] studied the Poisson stability (in particular, periodicity; Bohr almost periodicity; recurrence in the sense of Birkhoff; almost recurrence in the sense of Bebutov; Levitan almost periodicity) of solutions for the scalar equation (1) with f being monotone with respect to x and Poisson stable in $t \in \mathbb{R}$ (uniformly with respect to x on every compact subset of $\mathbb{R}$), that is, he generalized Opial's resut to the Poisson stable differential equation (1).

Cheban [1] considered the Poisson stable solutions for the scalar equation (1) with arbitrary Poisson stable (with respect to time t) f and without the monotonicity assumption for f.

In the framework of monotone cocycles or nonautonomous dynamical systems the problem of almost periodicity and almost automorphy of solutions for (1) in the general case (both finite and infinite dimensional cases) was studied in the works of Shen and Yi [8], Jiang and Zhao [2] and Novo et al. [3].

The aim of the paper under review is to study the existence of Poisson stable (e.g., stationary; periodic; quasi-periodic; Bohr/Levitan almost periodic; almost automorphic; almost recurrent, etc) solution of (1) in both finite and infinite dimensional cases when (1) generates a monotone cocycle, which can be achieved, say, if f is quasi-monotone.

Before stating the main results, we need to do some preparation.

Assume that X and Y are metric spaces and the metrics are denoted by ρ for simplicity. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_- = (-\infty, 0]$. For a given dynamical system $(X, \mathbb{R}, \pi)$ and a point $x \in X$, the *trajectory* and *semi-trajectory* are defined as $\Sigma_x := \{\pi(t,x) \colon t \in \mathbb{R}\}$ and $\Sigma_x^+ := \{\pi(t,x) \colon t \in \mathbb{R}_+\}$, respectively. We call the mapping $\pi(\cdot, x) \colon \mathbb{R} \to X$ the *motion* through x at the moment $t = 0$.

Let M be a subset of X. The ω*-limit set* of M is defined by

$$\omega(M) := \bigcap_{t\geq 0} \overline{\bigcup \{\pi(\tau, M) \colon \tau \geq t\}};$$

for a singleton set, we also write $\omega(x)$ or ω_x for $\omega(\{x\})$ for simplicity.

Let $\mathbb{T}_1 \subset \mathbb{T}_2$ be two subsemigroups of the group $\mathbb{R}$, $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ be two autonomous dynamical systems and $h : X \to Y$ be a homomorphism from $(X, \mathbb{T}_1, \pi)$ to $(Y, \mathbb{T}_2, \sigma)$ (that is, h is continuous and surjective, $h(\pi(t, x)) = \sigma(t, h(x)))$ for all $t \in \mathbb{T}_1$ and $x \in X$. Then the triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is called a *nonautonomous dynamical system* (NDS for short).

Let $(Y, \mathbb{R}, \sigma)$ be a two-sided dynamical system on Y and E be a metric space. A triplet $\langle E, \phi, (Y, \mathbb{R}, \sigma) \rangle$ is said to be a *cocycle* on the state space (or fibre) E with the base $(Y, \mathbb{R}, \sigma)$ if the mapping $\phi \colon \mathbb{R}_+ \times Y \times E \to E$ satisfies the following conditions:

(1) $\phi(0, u, y) = u$ for all $u \in E$ and $y \in Y$;
(2) $\phi(t + \tau, u, y) = \phi(t, \phi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{R}_+$, $u \in E$ and $y \in Y$;
(3) the mapping ϕ is continuous.

Let $\langle E, \phi, (Y, \mathbb{R}, \sigma) \rangle$ be a cocycle on E, $X := E \times Y$ and π be a mapping from $\mathbb{R}_+ \times X$ to X defined by $\pi := (\phi, \sigma)$, that is, $\pi(t, (u, y)) = (\phi(t, u, y), \sigma(t, y))$ for all $t \in \mathbb{R}_+$ and $(u, y) \in E \times Y$. The triplet $(X, \mathbb{R}_+, \pi)$ is called a *skew-product dynamical system*.

Let $\langle E, \phi, (Y, \mathbb{R}, \sigma) \rangle$ be a cocycle, $(X, \mathbb{R}_+, \pi)$ be the associated skew-product dynamical system with $X = E \times Y$ and $\pi = (\phi, \sigma)$, and $h \colon X \to Y$ be the natural projection to the second coordinate. Then the triplet $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ is an NDS.

Let (X, h, Y) be a fiber space, i.e., X and Y be two metric space and $h \colon X \to Y$ be a homomorphism. A set $M \subset X$ is said to be *conditionally precompact* if its intersection with the preimage of any precompact subset $Y' \subset Y$, i.e., the set $h^{-1}(Y') \cap M$, is a precompact subset of X. A set M is called *conditionally compact* if it is closed and conditionally precompact.

Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ be an NDS. A subset $A \subset X$ is said to be *(positively) uniformly stable* if for arbitrary $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, a) < \delta$ ($a \in A, x \in X$ and $h(a) = h(x)$) implies $\rho(\pi(t, x), \pi(t, a)) < \varepsilon$ for any $t \geq 0$. In particular, a point $x_0 \in X$ is called *uniformly stable* if the singleton set $\{x_0\}$ is so.

Section 3 of this paper is dedicated to study the structure of the ω-limit set of noncompact semi-trajectories for autonomous and nonautonomous dynamical systems. The main result of this section is as follows.

Theorem 1. *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ be an NDS with the following properties:*

(1) *there exists a point $x_0 \in X$ such that the positive semi-trajectory $\Sigma^+_{x_0}$ is conditionally precompact;*

(2) *the set ω_{x_0} is positively uniformly stable.*

Then all motions on ω_{x_0} can be extended uniquely to the left and hence the one-sided dynamical system $(X, \mathbb{R}_+, \pi)$ generates on ω_{x_0} a two-sided dynamical system $(\omega_{x_0}, \mathbb{R}, \pi)$.

In the section 4, the authors gave a survey of different classes of Poisson stable motions, Shcherbakov's principle of comparability of motions by their character of recurrence and some generalizations of this principle.

Section 5 of this paper is dedicated to study of abstract monotone nonautonomous dynamical system. First we need the following definition. For a given ordered bundle (X, h, Y), an NDS $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ is said to be *monotone* (*strictly monotone*, respectively) if $x_1 \le x_2$ ($x_1 < x_2$, respectively) implies $\pi(t, x_1) \le \pi(t, x_2)$ ($\pi(t, x_1) < \pi(t, x_2)$, respectively) for any $t > 0$.

For a given NDS $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$, let $S \subset X$ be a nonempty ordered subset possessing the following properties:

(1) $h(S) = Y$;

(2) S is positively invariant with respect to π, that is, $\langle (S, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ an NDS.

The authors used the following assumptions.

(C1) For every conditionally compact subset K of S and $y \in Y$ the set $K_y = h^{-1}(y) \cap K$ has both infimum $\alpha_y(K)$ and supremum $\beta_y(K)$.

(C2) For every $x \in S$, the semi-trajectory $\{\pi(t, x) : t \ge 0\}$ is conditionally precompact and its ω-limit set $\omega_x = \bigcap_{t\ge 0} \overline{\bigcup\{\pi(\tau, x) : \tau \ge t\}}$ is positively uniformly stable.

(C3) The NDS $\langle (S, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ is monotone.

(C4) Under the condition (C1), both $\alpha_y(K)$ and $\beta_y(K)$ belong to K_y for any $y \in Y$.

The authors proved the following auxiliary lemma.

Lemma 2. *Assume that (C1)-(C3) hold. For a given point $x_0 \in S$, let $K := \omega_{x_0}$ and $y_0 := h(x_0)$. Then*

(1) *if $q \in \omega_q \subset \omega_{y_0}$, $\alpha_q := \alpha_{y_0}(K)$, $K^1 := \omega_{\alpha_q}$, then the set $K^1_q := \omega_{\alpha_q} \cap h^{-1}(q)$ ($\omega_{\beta_q} \cap h^{-1}(y_0)$, respectively) consists of a single point γ_q (δ_q, respectively);*

(2) *let γ_q and δ_q be as in (1). Then we have $\gamma_q \le \alpha_q \le \beta_q \le \delta_q$.*

The main results of section 5 are as follows which give sufficient conditions for the existence of compatible and strongly compatible motions.

Theorem 3. *(Comparability). Assume that (C1)-(C3) hold. For a given* $x_0 \in S$*, let* $y_0 := h(x_0)$*. If* $y_0 \in \omega_{y_0}$*, i.e.* y_0 *is positively Poisson stable, then the point* γ_{y_0} *(*δ_{y_0}*, respectively) is comparable with* y_0 *by character of recurrence and*

$$\lim_{t\to+\infty} \rho(\pi(t,\alpha_{y_0}),\pi(t,\gamma_{y_0})) = 0 \ \ (\lim_{t\to+\infty} \rho(\pi(t,\beta_{y_0}),\pi(t,\delta_{y_0})) = 0 \textit{ respectively}).$$

Theorem 4. *(Strong Comparability). Assume that (C1)-(C3) hold,* $x_0 \in S$ *and* $y_0 = h(x_0)$ *is strongly Poisson stable. Then the point* γ_{y_0} *(*δ_{y_0}*, respectively) is strongly comparable with* y_0 *by character of recurrence and*

$$\lim_{t\to+\infty} \rho(\pi(t,\alpha_{y_0}),\pi(t,\gamma_{y_0})) = 0 \ \ (\lim_{t\to+\infty} \rho(\pi(t,\beta_{y_0}),\pi(t,\delta_{y_0})) = 0 \textit{ respectively}).$$

Using the above two theorems and Shcherbakov's comparability principle of motions by character of recurrence, the authors obtained a series of result of the existence of stationary motions, which are stated as follows.

Corollary 5. *Under (C1)-(C3), if the point* y_0 *is stationary (*τ*-periodic, Levitan almost periodic, almost recurrent, Poisson stable, respectively), then*

(1) *the point* γ_{y_0} *has the same recurrent property as* y_0*;*
(2) *the point* α_{y_0} *is asymptotically stationary (asymptotically* τ*-periodic, asymptotically Levitan almost periodic, asymptotically almost recurrent, asymptotically Poisson stable, respectively).*

Corollary 6. *Under (C1)-(C3), if the point* y_0 *is quasi-periodic (Bohr almost periodic, almost automorphic, Birkhoff recurrent, respectively), then*

(1) *the point* γ_{y_0} *(*δ_{y_0}*, respectively) has the same recurrent property as* y_0*;*
(2) *the point* α_{y_0} *(*β_{y_0}*, respectively) is asymptotically quasi-periodic (asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically Birkhoff recurrent, respectively).*

If in addition, y_0 *is Lagrange stable, then the above items (1) and (2) also hold for pseudo periodic and pseudo recurrent cases.*

The authors also gave some sufficient conditions which imply the convergence of all motions to Poisson stable one. This kind of convergence is fundamental in classical monotone dynamics.

Theorem 7. *Assume that (C1)-(C4) hold. For a given $x_0 \in S$, let $y_0 = h(x_0)$. If $y_0 \in \omega_{y_0}$, i.e. y_0 is positively Poisson stable, then the following statements hold:*

(1) *the point $\gamma_{y_0} \in \omega_{x_0}$;*
(2) *the point γ_{y_0} is comparable with y_0 by character of recurrence;*
(3) $\lim_{t\to+\infty} \rho(\pi(t, x_0), \pi(t, \gamma_{y_0})) = 0$.

The same result holds for δ_{y_0}, ie., (1)-(3) hold with γ_{y_0} replaced by δ_{y_0}.

Theorem 8. *Assume that (C1)-(C4) hold. For a given $x_0 \in S$, let $y_0 = h(x_0)$. If y_0 is strongly Poisson stable, then the following statements hold:*

(1) *the point $\gamma_{y_0} \in \omega_{x_0}$;*
(2) *the point γ_{y_0} is strongly comparable with y_0 by character of recurrence;*
(3) $\lim_{t\to+\infty} \rho(\pi(t, x_0), \pi(t, \gamma_{y_0})) = 0$*;*
(4) *the point γ_{y_0} has the same recurrent property as y_0 and the point x_0 has the same asymptotically recurrent property as α_{y_0}.*

The same result holds for δ_{y_0}, i.e., (1)-(4) hold with γ_{y_0} replaced by δ_{y_0}.

In section 6, the authors applied the abstract results obtained in sections 3 and 5 to study different classes of Poisson stability of solutions for monotone differential equations (ODEs, FDEs and parabolic PDEs). In this way they obtained a series of new results (some of them coincide with the well-known results).

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