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On involutions in symmetric groups and a conjecture of Lusztig

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Let (W, S) be a fixed Coxeter system with length function $\ell : W \to \mathbb{N}$. Let $\leq$ be the Bruhat partial ordering on W. Let $*$ be a fixed automorphism of W with order 2 and such that $s^* \in S$ for any $s \in S$.

Define the set of twisted involutions

$$I_* = \{w \in W \mid w^* = w^{-1}\}.$$

Let v be an indeterminate over $\mathbb{Z}$ and $u := v^2$. Let $\mathcal{H}_u$ be the Iwahori-Hecke algebra associated to (W, S) with Hecke parameter u^2 and defined over $\mathcal{A} = Z[v, v^{-1}]$. Let $\{T_w, w \in W\}$ be the standard basis of $\mathcal{H}_u$.

Lusztig (and with Vogan in special cases where W is a Weyl group or an affine Weyl group) have shown that the space spanned by I_* was naturally endowed with a module structure of the Hecke algebra of (W, S) with two distinguished bases, which can be viewed as twisted analogues of the well-known standard basis (a-basis)and Kazhdan-Lusztig basis(A-basis), as described in the following.

Let M be the free $\mathcal{A}$-module with basis $\{a_w \mid w \in I_*\}$. In [3, 5] Lusztig and Vogan proved the existence of the following $\mathcal{H}_u$-module structure:

$$T_s a_w = \begin{cases} u a_w + (u+1) a_{sw} & \text{if } sw = ws^* > w \\ (u^2 - u - 1) a_w + (u^2 - u) a_{sw} & \text{if } sw = ws^* < w \\ a_{sws^*} & \text{if } sw \neq ws^* > w \\ (u^2 - 1) a_w + u^2 a_{sws^*} & \text{if } sw \neq ws^* < w \end{cases}$$

$(\forall s \in S)$.

For each $w \in I_*$, they also proved further that there is a unique element

$$A_w = v^{-\ell(w)} \sum_{y \in I_*; y \leq w} P^{\sigma}_{y,w} a_y \in \underline{M}$$

$\overline{A_w} = A_w P^{\sigma}_{w,w} = 1$. If $y \in I_*, y < w$, then $degP^{\sigma}_{y,w} \leq \frac{1}{2}(\ell(w) - \ell(y) - 1)$.

The polynomials $P^{\sigma}_{y,w}$ can be viewed as a twisted analogue of the well-known Kazhdan-Lusztig polynomial $P_{y,w}$ of (W, S).

Lusztig [2] defined

$$X_{\emptyset} = \sum_{x \in W, x^* = x} u^{-\ell(x)} T_x$$

Let $\mathbb{Q}(u)$ be the field of rational functions on u. Set $\mathcal{H}^{\mathbb{Q}(u)} := \mathbb{Q}(u) \otimes \mathcal{H}_u$. Lusztig proposed the following conjecture:

There is a unique isomorphism of $\mathbb{Q}(u)$-modules $\eta : \mathbb{Q}(u) \otimes M \cong \mathcal{H}^{\mathbb{Q}(u)} X_{\emptyset}$ such that $a_1 \mapsto X_{\emptyset}$.

The main result of this paper is a proof of this conjecture in the case when $* = id_W$ and W is the symmetric group S_n on n letters (i.e., the Weyl group of type A_{n-1}) for any $n \in \mathbb{N}$.

As a byproduct of this paper, the authors showed that any two reduced I_*-expressions for an involution in S_n can be transformed into each other through a series of braid I_*-transformations, which can be viewed as a twisted analogue of a well-known classical fact of Matsumoto which said that any two reduced expressions for an element in S_n can be transformed into each other through a series of braid transformations.

This work was generalized by the authors in their following serial paper [1], where they considered the case when $* = id$ and W were the Weyl groups of types B_n and D_n. In [1] they identified a set of basic braid I_*-transformations which span and preserve the sets of reduced I_*-expressions for any involution.

Note that these generalizations are non-trivial in the sense that the basic braid I_*-transformations for the Weyl group of types B_n and D_n which they identified contain not only the usual basic braid transformations plus some natural right end transformations but also one extra transformations which do not directly related to the usual basic braid transformations. This is a new phenomenon which does not happen in type A case.

Later on, we also noticed that Lusztig had proved his conjecture (in [4]) for any Coxeter group and any $*$ by using a completely different argument.

Despite this fact, it is still seminal and interesting in itself to generalize Matsumoto's result for reduced I_*-expressions of involutions to Weyl groups of arbitrary types (including type A).

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