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A variant of Horn's problem and the derivative principle
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A. Horn's problem address the following problem. Given are $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, $\mu = (\mu_1, \mu_2, \dots, \mu_n)$, $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \mathbb{R}^n$. Under what conditions do there exists $n \times n$ Hermitian matrices A and B so that

$$\sigma(A) = \lambda, \sigma(B) = \mu, \text{ and } \sigma(A + B) = \nu? \tag{1}$$

Here $\sigma(M)$ denotes the n tuple of eigenvalues of the $n \times n$ matrix M for which we always assume that the tuples are ordered to satisfy

$$\lambda_1 \geq \cdots \geq \lambda_n, \ \mu_1 \geq \cdots \geq \mu_n, \text{ and } \nu_1 \geq \cdots \geq \nu_n.$$

In 1962, Alfred Horn [5] provided a list of inequalities, all having the form,

$$\sum_{k \in K} \nu_k \leq \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j, \tag{2}$$

for certain subsets I, J, K of $\{1, 2, \dots, n\}$ with the same cardinality r, with $r < n$. More specifically, Horn defined sets T_r^n of triples (I, J, K) by the following inductive procedure. Set

$$U_r^n = \{(I, J, K) \mid \sum i + \sum j = \sum k + r(r+1)/2\}.$$

When $r = 1$, set $T_1^n = U_1^n$. Otherwise

$$T_r^n = \{(I, J, K) \in U_r^n \mid \text{for all } p < r \text{ and all } (F, G, H) \in T_p^r,$$

$$\sum_{f \in F} i_f + \sum_{g \in G} j_g \leq \sum_{h \in H} k_h + p(p+1)/2\}.$$

Horn conjectured that a triple (λ, μ, ν) is a solution of Problem (1) if and only if

$$\sum_{k \in \bar{K}} \nu_k \geq \sum_{i \in \bar{I}} \lambda_i + \sum_{j \in \bar{J}} \mu_j, \tag{3}$$

where $\bar{I}$, $\bar{J}$ and $\bar{K}$ are the complements of I, J and K in $\{1, 2, \ldots, n\}$, hold for every (I, J, K) in T_r^n for all $r < n$. (2) and (3) are equivalent due to the trace equality.

A. Horn's problem had been open for a couple of decades. The key for proving A. Horn's problem was another long-standing conjecture, the saturation conjecture, saying that

$$C_{\lambda,\mu}^{\nu} \neq 0 \Longleftrightarrow C_{N\lambda,N\mu}^{N\nu} \neq 0$$

where both $C_{\lambda,\mu}^{\nu}$ and $C_{N\lambda,N\mu}^{N\nu}$ are Littlewood-Richardson coefficients which can be defined via a number of combinatorial objects, such as hives, honeycomb models and tableau. The saturation conjecture was proved by Allen Knutson and Terence Tao [7] in 1999, which was considered as a major breakthrough in a number of fields because of the prevalence of Littlewood-Richardson coefficients. In combinatorics, they appear in the theory of symmetric functions. The Schur symmetric functions form a linear basis of the ring of symmetric functions, and the Littlewood-Richardson coefficients express the multiplication rule,

$$S_\lambda S_\mu = \sum_{\nu} C_{\lambda\mu}^{\nu} S_\nu.$$

In the representation theory of the general and special linear groups, the characters of the irreducible polynomial representations of $GL_n(\mathbb{C})$ are Schur functions in appropriate variables. As such, Littlewood-Richardson coefficients $C_{\lambda\mu}^{\nu}$ give the multiplicity with which the irreducible representation V_ν of $GL_n(\mathbb{C})$ appears in the tensor product of the irreducible representations V_λ and V_μ:

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu} C_{\lambda\mu}^{\nu} V_\nu.$$

Littlewood-Richardson coefficients also appear in algebraic geometry: Schubert classes form a linear basis of the cohomology ring of the Grassmannian, and the Littlewood-Richardson coefficients again express the multiplication rule:

$$\sigma_\lambda \sigma_\mu = \sum_\nu C^\nu_{\lambda\mu} \sigma_\nu.$$

The book [2] is a good source for all these aspects of Littlewood-Richardson coefficients and the papers [3] and [8] provide good overviews on the history and the solution of the saturation conjecture and A. Horn's problem.

Another important ingredient for proving A. Horn's problem was the connection between A. Horn's problem and the saturation conjecture which was given by Heckman [4] implicitly and Alexander Klyachko explicitly [6] who used geometric invariant theory to reduce A. Horn's problem to the saturation conjecture in 1998. So A. Horn's problem was proven completely via a combination of the result by Klyacho and those of Knutson and Tao.

In 2014, Matthias, Doran, Kousidis and Walter [1] provide an effective method, rooted in symplectic geometry, to compute the joint probability distribution of the eigenvalues of its one-body reduced density matrices. As a corollary, by taking the distributions support, which is a convex moment polytope, they recover a complete solution to the one-body quantum marginal problem which subsumes A. Horn's problem, and then obtain the probability distribution by reducing to the corresponding distribution of diagonal entries, which is determined algorithmically.

In the paper under review, Zhang and Xiang obtain the probability distribution density of the eigenvalues of sum of two Hermitian matrices with prescribed spectra via the result in [1]. Moreover, the support for such probability distribution density function is just Horn's polytope, determined by Horn's inequalities.

The result in [1] is not limited to two particles, so one may wonder if it can be applied to the eigenvalues of the sum of more than two Hermitian matrices.

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