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Yang, Yunyan (杨云雁); Zhu, Xiaobao (朱晓宝)

A Trudinger-Moser inequality for a conical metric in the unit ball

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评论员：周春琴 (上海交通大学，上海)

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In the paper under review, the authors prove the existence of extremals for a Trudinger-Moser inequality, namely,

$$T_0 := \sup\{\int_{\mathbb{B}} e^{\alpha_N(1+\frac{p}{N}\beta)|u|^{\frac{N}{N-1}}}|x|^{p\beta}dx : u \in W_0^{1,N}(\mathbb{B}), ||u||_\alpha \leq 1\} < +\infty, \qquad (1)$$

whenever $p > 1$, $\beta \geq 0$ and $\alpha < (1+\frac{p}{N}\beta)^{N-1+\frac{N}{p}}\lambda_p(\mathbb{B})$. Here, $\mathbb{B}$ is the unit ball in $\mathbb{R}^N (N \geq 2)$, $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$, ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$,

$$\lambda_p(\mathbb{B}) = \inf\{\int_{\mathbb{B}} |\nabla u|^N dx : u \in W_0^{1,N}(\mathbb{B}), \int_{\mathbb{B}} |u|^p dx = 1\},$$

and

$$||u||_\alpha = \int_{\mathbb{B}} |\nabla u|^N dx - \alpha(\int_{\mathbb{B}} |u|^p |x|^{p\beta} dx)^{\frac{N}{p}}.$$

The inequality (1) can be seen as a inequality on the unit ball with a conical metric, since there is a geometric meaning of the term $|x|^{p\beta}dx$. Precisely, letting $g_0 = dx_1^2 + \cdots + dx_N^2$ be the standard Euclidean metric, and defining a metric $g = |x|^{p\beta}g_0$ for $x \in \mathbb{B}$, $(\mathbb{B}, g)$ is a conical manifold with the volume element $dv_g = |x|^{p\beta}dx$. Moreover, $|\nabla u|^N dx = |\nabla_g u|^N dv_g$.

When $p = N$, $-1 < \beta < 0$ and $\alpha = 0$, the inequality (1) was shown by Adimurthi and Sandeep [1]. When $p = N = 2$, $\beta \geq 0$ and $\alpha = 0$, the inequality

(1) was shown by de Figueiredo [2]. Therefore, in this paper, the authors have generalized the previous results to the more general cases.

When $\beta > 0$, the situation is subtle. It is worthy to mention that authors give an example to show that the inequality is not valid for the case of $p = N$, $\alpha = 0$ and $\beta > 0$, namely, the supremum for $\beta \geq 0$

$$\sup\{\int_{\Omega} e^{\alpha_N(1+\beta)|u|^{\frac{N}{N-1}}}|x|^{N\beta}dx : u \in W_0^{1,N}(\Omega), ||u||_{W_0^{1,N}(\Omega)} \leq 1\}$$

is not finite.

The general strategy employed is based on a change of variables and blow-up analysis. The blow up method is standard and important in Geometry Analysis. The main steps are:

1. Notice that $\mathbb{B}$ is a unit ball, which is centered at the origin. Let ϱ be a set of all radially symmetric functions. Then prove in the case of $p = N$, $\beta \geq 0$ and radially symmetric functions u that
$$\sup\{\int_{\mathbb{B}} e^{\alpha_N(1+\beta)|u|^{\frac{N}{N-1}}}|x|^{N\beta}dx : u \in W_0^{1,N}(\mathbb{B}) \cap \varrho, ||u||_{W_0^{1,N}(\mathbb{B})} \leq 1\} < +\infty,$$
and the supremum can be attained. The strategy employed is based on a change of variables.
2. By using the standard blow up procedure to prove in the case of $p > 1$ and $\beta = 0$ that
$$\sup\{\int_{\mathbb{B}} e^{\gamma|u|^{\frac{N}{N-1}}}dx : u \in W_0^{1,N}(\mathbb{B}), ||u||_{\alpha} \leq 1\} < +\infty$$
for any $\gamma \leq \alpha_N$, and the supremum can be attained.
3. Using a change of variables again and putting step 1. and step 2. together to prove (1).

References

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