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A Casselman-Osborne theorem for rational Cherednik algebras
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During the past two decades, the Dirac operator has played an important role in representation theory. In 1990s, D. A. Vogan introduced the notion of Dirac operator and Dirac cohomology for admissible $(\mathfrak{g}, K)$-modules of real reductive Lie groups and conjectured that the Dirac cohomology (if nonzero) of an irreducible $(\mathfrak{g}, K)$-module M determines the infinitesimal character of M, which was verified by Huang and Pandžić [4]. Later on, the notion of Dirac operator was generalized to different algebraic settings, e.g., affine Lie algebras, Lie superalgebras, graded affine Hecke algebra and symplectic reflection algebras including rational Cherednik algebras.

Let $\mathfrak{h}$ be a complex vector space, W a complex reflection group, i.e., a subgroup of $GL(\mathfrak{h})$ generated by some complex reflections s fixing a hyperplane $H_s \subseteq \mathfrak{h}$, and $\mathcal{R}$ the subset of all complex reflections in W. The natural pairing $\langle\ ,\ \rangle$ between $\mathfrak{h}$ and $\mathfrak{h}^*$ is W-invariant.

Let $\alpha_s \in \mathfrak{h}^*$ be a nonzero vector such that $\langle y, \alpha_s\rangle = 0$ for all $y \in H_s$. And define $\alpha_s^\vee \in \mathfrak{h}$ similarly. Then we have

Definition 1 ([2, 3])**.** *For any W-invariant function $c : \mathcal{R} \to \mathbb{C}$, $t \in \mathbb{C}$, the rational Cherednik algebra $\boldsymbol{H}_{t,c}$ associated to $\mathfrak{h}$ and W is defined as the quotient of $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes \mathbb{C}[W]$ by the relation*

$$[y,x] = t\langle y,x\rangle - \sum_{s\in\mathcal{R}} c(s)\frac{\langle y,\alpha_s\rangle\langle\alpha_s^\vee, x\rangle}{\langle\alpha_s^\vee,\alpha_s\rangle}s,$$

for all $y \in \mathfrak{h}$ and $x \in \mathfrak{h}^*$

Set $V = \mathfrak{h} \oplus \mathfrak{h}^*$. Let $C(V)$ be the Clifford algebra of V with respect to the natural symmetric pairing on V. Fix a basis $x_1, \ldots, x_n$ of $\mathfrak{h}^*$ and the dual basis $y_1, \ldots, y_n$ of $\mathfrak{h}$, Ciubotaru [1] introduced the Dirac operator in $\mathbf{H}_{t,c} \otimes C(V)$ as

$$D = D_x + D_y,$$

where $D_x = \sum_i x_i \otimes y_i$, $D_y = \sum_i y_i \otimes x_i$. The Dirac cohomology for an $\mathbf{H}_{t,c}$ module M is defined by

$$H_D(M) = \ker D / \operatorname{im} D,$$

where D is regarded as a homomorphism of $M \otimes S$ for a spinor module S of $C(V)$.

For a $\star$-unitary module M of $\mathbf{H}_{t,c}$, D is skew-adjoint, since $D_x^* = -D_y$ and $D_y^* = -D_x$. Therefore,

$$H_D(M) = \ker D = \ker D^2.$$

Furthermore, the authors got the following Hodge decomposition.

Theorem 2 (Ref. Theorem 5.4)**.** *(a)* $M \otimes S = \ker D \oplus \operatorname{im} D_x \oplus \operatorname{im} D_y$.
(b) $\ker D_x = \ker D \oplus \operatorname{im} D_x$, $\ker D_y = \ker D \oplus \operatorname{im} D_y$.

Inspired by the work in [5], where the main results is, under some assumptions, that the Dirac cohomology of a unitary $(\mathfrak{g}, K)$-module V is isomorphic to the $\bar{\mathfrak{u}}$-cohomology of V and the $\mathfrak{u}$-homology of V up to appropriate modular twists, the authors defined (co)homology of an $\mathbf{H}_{t,c}$-module analogously (Definitions 2.3 and 2.5).

To be more precise, for any $\mathbf{H}_{t,c}$-module M, the p^{th} $\mathfrak{h}^*$-cohomology group $H^p(\mathfrak{h}^*, M)$ is identified with the p^{th} cohomology group of the cochain complex

$$0 \to \operatorname{Hom}_{\mathfrak{h}^*}(\wedge^0\mathfrak{h}^*, M) \xrightarrow{d_0} \cdots \xrightarrow{d_{n-1}} \operatorname{Hom}_{\mathfrak{h}^*}(\wedge^n\mathfrak{h}^*, M) \to 0;$$

and the p^{th} $\mathfrak{h}$-homology group $H_p(\mathfrak{h}, M)$ is identified with the p^{th} homology group of the chain complex

$$0 \to M \otimes \wedge^n\mathfrak{h} \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} M \to 0.$$

Then we have

Proposition 3 (Ref. Proposition 2.12)**.** *Let $\widetilde{W}$ be a spin double cover of W. There are $\widetilde{W}$-module isomorphisms:*

$$\ker D_x/\operatorname{im} D_x \cong H^*(\mathfrak{h}^*, M) \otimes \chi, \quad \textit{and} \quad \ker D_y/\operatorname{im} D_y \cong H_*(\mathfrak{h}, M) \otimes \chi,$$

where χ is a genuine one-dimensional $\widetilde{W}$-module (see Proposition 2.9).

The main results of the paper under review are the following.

Theorem 4 (Ref. Theorems 4.2 and 5.1)**.** *If M is an $\boldsymbol{H}_{t,c}$-module so that D^2 acts semisimply on $M \otimes S$, then we have the following injective homomorphisms of $\widetilde{W}$-modules:*

$$H_D(M) \hookrightarrow H^*(\mathfrak{h}^*, M) \otimes \chi, \quad H_D(M) \hookrightarrow H_*(\mathfrak{h}, M) \otimes \chi.$$

If, furthermore, M is a $*$-unitary $\boldsymbol{H}_{t,c}$-module, then the above injections are isomorphisms.

The main ingredient of the paper under review is essentially the same as that in [5]. It is natural to ask whether the same method can be applied to other algebras, say, Lie algebras with triangular decomposition and other algebras with similar structures.

References

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