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Genus-one mirror symmetry in the Landau-Ginzburg model
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Enumerative geometry is an ancient subject. It concerns about counting geometric objects satisfying a certain number of geometric conditions. For example, there is a unique line passing through 2 points in the plane, and there is a unique conic passing through 5 points in the plane. In general, one can ask how many rational algebraic curves of degree d pass through $3d-1$ general points in the plane. A solution to this question can be obtained by studying the so-called *Gromov-Witten theory*. Gromov-Witten theory is a curve counting theory heavily influenced by string theory. In string theory, the six "unseen" dimensions of spacetime are modelled by Calabi-Yau manifolds of complex dimension 3 (Calabi-Yau threefolds). As strings propagate through the spacetime, they sweep out 2-dimensional manifolds (worldsheets) in a Calabi-Yau threefold. Therefore, it is an interesting question to count the number of algebraic curves on a Calabi-Yau threefold. These numbers are called Gromov-Witten (GW) invariants. If the string theory describes the real world, we can detect these numbers via physical experiments!

One of the simplest examples of Calabi-Yau threefolds is the Fermat quintic threefold X. It is defined by the zero loci of the single degree-five homogeneous polynomial

$$W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$$

inside $\mathbb{CP}^4$, where x_i are the homogeneous coordinates. It was a very challenging question in the 80s' to compute the degree-d, genus-zero GW invariants on X, i.e., the number of degree-d rational algebraic curves on X. Mathematicians were able to compute these numbers only for $d \leq 5$. In 1991, Physicists Candelas, de la Ossa, Green, and Parkes gave formulas for these numbers for all d in [4], using string theory and mirror symmetry. In mathematics, this conjecture was confirmed by Givental [24] and Lian-Liu-Yau [38] in the 90s'.

The computation of higher-genus GW invariants is much more difficult than the genus-zero case. The genus-one GW invariants of the quintic were computed by Zinger, after a great amount of hard work [48]. A natural idea to obtain an effective method of computing higher-genus GW invariants is by finding an equivalent model of the GW theory of the quintic Calabi-Yau (CY) threefolds in which the computations are easier. Note that the Fermat quintic polynomial $W : \mathbb{C}^5 \to \mathbb{C}$ has an isolated critical point (singularity) at the origin. In physics, one can associate to it the so-called *Landau-Ginzburg model*. It was argued in [44, 45] that there should be a Landau-Ginzburg/Calabi-Yau (LG/CY) correspondence connecting CY model to the LG singularity model. A candidate quantum theory of singularities was constructed by Fan-Jarvis-Ruan [20, 21], and it is now called the Fan–Jarvis–Ruan–Witten (FJRW) theory. From the analytic point of view, the moduli spaces in FJRW theory parametrize solutions to the *Witten equation*

$$\bar{\partial} u_i + \overline{\frac{\partial W}{\partial u_i}} = 0.$$

From the algebraic point of view, the moduli spaces parametrize W-curves, and the FJRW theory generalizes the theory of r-spin curves [1, 34, 33, 35].

The genus-zero FJRW invariants were computed by Chiodo-Ruan [12]. They compared the genus-zero mirror formulas of the GW theory and FJRW theory of the quintic and provided a mathematically precise statement of the genus-zero LG/CY correspondence. The genus-zero result in [12] was formulated using Givental's symplectic formalism [26]. This provided a hint at how to formula a higher-genus correspondence. In the paper under review, the authors studied the genus-one FJRW invariants and verified the genus-one mirror symmetry formula conjectured by Huang-Klemm-Quackenbush [32]. In the sequel [29], the authors proved the genus-one LG/CY correspondence by using the genus-one mirror formulas proved in the paper under review and in [48].

Before introducing the main results of the paper under review, let me briefly explain the difficulty in computing higher-genus GW and FJRW invariants. Both invariants are defined as intersections numbers over certain moduli spaces. In GW theory, the moduli spaces (of stable maps) are very singular in general. To obtain meaningful intersection numbers over these singular moduli spaces, Li-Tian [37] and Behrend-Fantechi [2] introduced the concept of virtual fundamental cycles. These cycles are difficult to compute except in the genus zero case. Zinger's method [48] is based on the desingularization of the genus-one moduli spaces. The desingularization is much harder when the genus $g \geq 2$ (see [31]), and so far, no one has been able to compute genus-two invariants via this method. In FJRW theory, the moduli spaces (of W-curves) are smooth. However, the virtual cycles are defined in a non-trivial fashion and difficult to compute when $g \geq 1$.

The main theorem of the paper under review is the following:

Theorem 1 ([30]).

$$\sum_{n>1} \frac{1}{n!} \langle (\tau\phi_1)^n \rangle_{1,n}^{\mathrm{w},\infty} = \log \left((I_0(t))^{-31/3} (1-(t/5)^5)^{-1/12} \tau'(t)^{-1/2} \right).$$

Here $\langle (\tau\phi_1)^n \rangle_{1,n}^{w,\infty}$ denote the genus-one FJRW invariants with n identical insertions $\tau\phi_1$, where $\tau = I_1(t)/I_0(t)$ is an explicit function (mirror map) determined by genus-zero FJRW invariants and ϕ_1 describes the multiplicity of orbifold points of the W-curves.

The proof consists of two parts:

- Step 1: Comparison between genus-one FJRW invariants and *twisted 5-spin invariants*. Let $\overline{\mathcal{M}}_{g,\vec{m}}^{1/5}$ denotes the moduli space of 5-spin curves. The obstruction theory is given by a two-term complex $R\pi_*\mathcal{L}^{\oplus 5}$ of coherent sheaves, where $\mathcal{L}$ is the universal line bundle and π is the projection from he universal curve. One can equip $\overline{\mathcal{M}}_{g,\vec{m}}^{1/5}$ with two different virtual fundamental cycles. On the one hand, we have the FJRW virtual cycle. On the other hand, we consider the $(\mathbb{C}^*)^5$-action on $R\pi_*\mathcal{L}^{\oplus 5}$ via scaling the five different copies of $\mathcal{L}$. The twisted virtual cycle is defined as the inverse equivariant Euler class of the two-term obstruction theory. The invariants defined by these two types of virtual cycles are labeled by $\star = \mathrm{w}$ and λ, respectively, in the paper under review. There is a good algorithm to compute twisted invariants (see the introduction to the second part of the proof).

  In the genus-zero case, one of the two terms in $R\pi_*\mathcal{L}^{\oplus 5}$ vanishes. Therefore, the FJRW virtual cycle coincides with the twisted virtual cycle. This fails in the higher-genus situation. When $g = 1$, the sublocus in the moduli space $\overline{\mathcal{M}}_{g,\vec{m}}^{1/5}$ where the obstruction theory fails to be a vector bundle parametrizes 5-spin curves with rational tails. One way to eliminate the locus of rational tails is via "wall-crossing" techniques. These techniques were first developed in the geometry setting [43, 16, 13, 15] and then in the LG setting [41]. In the moduli space $\overline{\mathcal{M}}_{g,\vec{m}}^{1/5}$, the marked points are "heavy" points. By crossing the walls, the authors obtained an alternate compactification $\overline{\mathcal{M}}_{g,\vec{m}}^{1/5,0}$ in which the orbifold curves do not have rational tails. Similarly, one can define FJRW-type virtual cycle and twisted virtual cycle over $\overline{\mathcal{M}}_{g,\vec{m}}^{1/5,0}$.

The authors proved the following genus-one wall-crossing formula

Theorem 2 ([30]). *For $\star = \mathrm{w}$ or λ, we have*

$$\sum_{n>1} \frac{1}{n!} \langle (\tau\phi_1)^n \rangle_{1,n}^{\star,\infty} = \log\left((I_0(t)) \langle \phi_0 \psi_1 \rangle_{1,1}^{\star,\infty} + \sum_{\delta>0} t^\delta \langle - \rangle_{1,0|\delta}^{\star,0}.\right.$$

Here $\langle (\tau\phi_1)^n \rangle_{1,n}^{\mathrm{w},\infty}$ (resp. $\langle (\tau\phi_1)^n \rangle_{1,n}^{\lambda,\infty}$) denotes genus-one n-pointed FJRW invariants (resp. twisted invariants) and $\langle - \rangle_{1,0|\delta}^{\mathrm{w},0}$ (resp. $\langle - \rangle_{1,0|\delta}^{\lambda,0}$) denotes the FJRW type (resp. twisted) invariants with light markings. Due to the absence of heaving markings and rational tails, the obstruction theory is given by a vector bundle and hence we have

$$\langle - \rangle_{1,0|\delta}^{\mathrm{w},0} = \langle - \rangle_{1,0|\delta}^{\lambda,0}.$$

The comparison between genus-one FJRW invariants and twisted 5-spin invariants follows from Theorem 2.

The strategy of proving Theorem 2 is by studying the localization relations on the master space introduced independently by Chang-Li-Li-Liu [9] and Fan-Jarvis-Ruan [22]. Following the detailed computations of Chang-Li-Li-Liu [8], the authors were able to reduce the proof of Theorem 2 to the proofs of a genus-zero LG-type wall-crossing result and a genus-one quasimap type wall-crossing result. The genus-zero wall-crossing result is obtained by adapting arguments analogous to those developed in [3, 19, 41]. The genus-one quasimap type wall-crossing result is obtained by using arguments developed in [39, 15].

- Step 2: Explicit computations of two genus-one one-point invariants and twisted 5-spin invariants using the Givental-Teleman quantization formula for semisimple CohFTs [25, 42]. Guo is a leading expert in Givental formalism and computing twisted invariants.

Let me conclude the review with some remarks on related work and further development. The idea of using an alternate compactification (stable quasimap theory) to compute genus-one GW invariants of the quintic was carried out in [36], which reproved the results of Zinger [48] and Popa [40]. The all-genus LG wall-crossing formula, generalizing Theorem 2, was proved by Zhou [47]. It was generalized to the setting of the gauged linear sigma model by Clader-Janda-Ruan [18]. On the geometric side, the all-genus quasimap wall-crossing formula was proved by Ciocan-Fontanine–Kim [14], Clader-Janda-Ruan [17] and in most generality by Zhou [46]. There is a great development in computing higher-genus GW invariants of quintic threefolds and proving their structures. One approach is the theory of Mixed-Spin-P (MSP) fields used in the paper under review. It was further developed and applied in [9, 8, 7, 6, 5]. Another approach is by using logarithmic compactification [11, 27, 28, 10]. Also, see a different approach in [23].

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