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The crepant transformation conjecture for toric complete intersections

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Assume that X_1 and X_2 are two smooth varieties, orbifolds, or Deligne-Mumford stacks, they are called K-equivalent if there exists a common Y and projective birational morphisms $f_1 : Y \to X_1$, $f_2 : Y \to X_2$ such that $f_1^* K_{X_1} = f_2^* K_{X_2}$. K-equivalence between X_1 and X_2 is also known as crepant transformation. The celebrated crepant transformation conjecture (CTC for short) states that two K-equivalent toric Deligne-Mumford stacks (or orbifolds) have isomorphic quantum cohomology rings (or F-manifold structures) after analytic continuation of quantum parameters in extended kahler moduli space. Since this conjecture connects Gromov-Witten theory and birational geometry, it inspires a lot of relevant work. Recently, this conjecture can be reformulated by Coates-Iritani-Tseng/Ruan using Givental's symplectic geometry formalism [1, 2]. That is to say, to each space X, we can associate it with vector space $\mathcal{H}(X) := \mathcal{H}^*_{CR,T}(X)((z^{-1}))$ plus some symplectic form Ω, then the Lagrangian cone $\mathcal{L} \subset \mathcal{H}(X)$ contains all information about T-equivariant genus zero Gromov-Witten invariants. Thus, determining the Givental cone $\mathcal{L}$ is equivalent to determining quantum product on $\mathcal{H}^*_{CR,T}(X)$. Correspondingly, the CIT/Ruan's version of Crepant Transformation Conjecture turns into proving the existance of $\mathbb{C}((z^{-1}))$-linear grading-preserving symplectic isomorphism $\mathbb{U}$ such that $\mathbb{U}(\mathcal{L}_1) = \mathcal{L}_2$ after analytic continuation.

This paper under review considers crepant transformation conjecture for the case of two toric varieties or complete intersections in toric varieties (GIT quotients of the form $\mathbb{C}^m//_\omega K$) which are connected by crepant transformation, where K is a complex torus group and ω is some stability condition. Here the K-equivalence (i.e. crepant transformation) between X_1 and X_2 is induced by a wall-crossing in the space of stability conditions. The main results are stated in the language of equivariant quantum connections for X_1 and X_2. They show that the equivariant quantum connections for toric varieties X_1 and X_2 (correspondingly, complete intersections) are globally gauge-equivalent after analytic continuations. The relation between gauge transformation θ and symplectic transformation $\mathbb{U}$ is $\mathbb{U} = L_1^{-1}\theta L_2$. The symplecticity of $\mathbb{U}$ follows from identification of θ with Fourier-Mukai transformation $\mathbb{FM}$, where $\mathbb{FM}$ is a derived equivalence and preserves Mukai pairings. The main tools for proof are mirror theorem for toric stacks and Mellin-Barnes method for analytic continuation of hypergeometric functions.

The paper under review is organized in the following way:

In section §2, the authors introduce some necessary notations (like equivariant Chen-Ruan cohomology, Gromov-Witten invariants, quantum cohomology, quantum connection, etc.) appearing in equivariant Gromov-Witten theory of manifolds/orbifolds and explain equivalent relations among torus-equivariant genus zero Gromov-Witten invariants, Givental's lagrangian cones, and equivariant quantum connections.

In section §3, the authors introduce equivariant integral structure which generalizes the integral structure for non-equivariant quantum cohomology [4], [5]. The equivariant integral structure is a $K_T^0(pt)-$ lattice in the space of flat sections for the equivariant quantum connection. The equivariant integral structure is described by T-equivariant characteristic classes (called Gamma class) which plays the role of square root of the Todd class. Furthermore, Similar structures are also explored in the case of Hilbert scheme of points in $\mathbb{C}^2$ [6] and toric Calabi-Yau 3-folds [7] separately.

In section §4, the authors explain how to see toric Deligne-Mumford stacks as GIT quotients and properties about GIT quotients. Especially, the notion of extended stacky fan [8] is introduced .

In section §5, the authors introduce wall-crossing in toric Gromov-Witten theory. A toric wall-crossing can be expressed as "a modification along a circuit" [9]. There are possibly three types of crepant toric wall-crossings: “flop”, “crepant resolution”, and “gerbe flop”. A global equivariant quantum connection can be constructed by applying Mirror theorem for toric Deligne-Mumford stacks to certain cohomology-valued hypergeometric function (called I-function). The corresponding analytic continuation between two equivariant quantum connections can be derived in terms of Mellin-Barnes transformation between two I-functions.

In section §6, the authors show that the global quantum connections are gauge-equivalent. The gauge equivalence is homogeneous of degree zero and preserves orbifold Poincare pairing. The Fourier-Mukai transformation between K-groups imply the analytic continuation between K-theoretic flat sections by setting Novikov variable to be 1.

In section §7, Crepant Transformation Conjecture for the cases of complete intersections has been studied in a similar fashion. The main tools are Mirror theorem for toric complete intersection and non-linear Serre duality.

In conclusion, the paper under review presents a fully equivariant proof of Crepant Transformation Conjecture for toric varieties and toric complete intersections at genus zero. The highlight is to show global gauge equivalence of equivariant quantum connections under analytic continuation in terms of equivariant Gamma-integral structure. Compared with Gonzalez-Woodward's work [10] (based on gauged Gromov-Witten theory) about more general wall-crossing formula for Gromov-Witten invariants under variation of GIT quotients, this paper provides a more refined approach to understanding relationship between genus zero Gromov-Witten theories.

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