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Whittaker-Shintani functions for general linear groups over p-adic fields
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The explicit construction of automorphic L-functions is an important topic in the study of automorphic representations. As indicated by the classical Hecke theory, this is usually done by constructing certain global zeta integrals which can present the required automorphic L-functions. In [10], Godement and Jacquet give an integral presentation of the standard L-function of an irreducible cupsidal automorphic representation of general linear groups GL_m, and this is extended to Rankin-Selberg L-functions of $\mathrm{GL}_m \times \mathrm{GL}_n$ by Jacquet, Piatetski-Shapiro and Shalika in [11].

The construction can also be extended to Rankin-Selberg L-functions of $\mathrm{GL}_m \times \mathbf{G}$, where $\mathbf{G}$ is a classical group. Such works are pioneered by Gelbart, Piatetski-Shapiro and Rallis ([4]), Ginzburg ([5]), Soudry ([19]) and Ginzburg-Rallis-Soudry ([8]) for generic cuspidal automorphic representations. For non-generic representations, the constructions are given by Ginzburg, Piatetski-Shapiro and Rallis in [7] for orthogonal groups, and by Ginzburg-Jiang-Rallis-Sourdy ([6]) and Shen ([18]) for symplectic groups and unitary groups. The constructions are extended to more general context by the works of Jiang and Zhang for quasi-split classical groups of Hermitian type ([12]), as well as their inner forms ([13]).

In the above works, a key step is to calculate the local integrals which belong to the Eulerian product of the global zeta integrals explicitly, at unramified places. These calculations will give the expected unramified local L-factors. For the case of generic representations, the Casselman-Shalika formula ([2]) for the Whittaker functions is used to calculate the unramified local integrals. And if the representations are non-generic, one needs to consider unramified representations which lie in certain local models, i.e., the local Bessel models or Fourier-Jacobi models. Then the calculation of the local integrals relies on *Whittaker-Shintani* functions, which is first introduced by Shintani and was used in the study of automorphic L-functions of Siegel and Jacobi cusp forms.

The Whittaker-Shintani functions are studied systematically in the works of Murase-Sugano ([16, 17]), Kato-Murase-Sugano ([15]) and Shen ([18]) in various cases. The techniques and results in these works are then used in the calculation of corresponding global zeta integrals presenting Rankin-Selberg L-functions (see [7], [12] and [20]). It is also worthwhile to mention that such calculations are also be used in the study of automorphic descent method and the related branching problems of automorphic representations (see [13] and [14]).

The paper in hand gives an explicit formula of the Whittaker-Shintani functions of Bessel type for general linear groups, extending the corresponding results in [17]. Let F be a p-adic field of characteristic zero with odd p. For simplicity, denote $G_m = \mathrm{GL}_m(F)$, and also $K_m = \mathrm{GL}_m(\mathcal{O})$, the maximal compact subgroup in G_m, here $\mathcal{O}$ is the ring of integers in F. Let $B_m = T_m \cdot U_m$ be the Borel subgroup of G_m consisting all upper-triangular elements, where T_m is the split torus contained in B_m, and U_m is the unipotent radical of B_m. Moreover, let δ_m be the modular character of B_m.

Let $N_\ell^{(m)}$ $(0 < m < \lfloor \frac{m}{2} \rfloor)$ be the unipotent subgroup of G_m consisting of elements of the form

$$u = \begin{pmatrix} a & b & x \\ & I_{m-2\ell} & c \\ & & d \end{pmatrix},$$

where $a, d \in U_\ell$. For a unramified additive character $\psi : F \longrightarrow \mathbb{C}^\times$, let $\psi_\ell^{(m)}$ be the character of $N_\ell^{(m)}$ given by

$$\psi_\ell^{(m)}(u) = \psi\left(\sum_{i=1}^{\ell-1}(a_{i,i+1} + d_{i,i+1}) + b_{\ell,m-2\ell} + c_{1,m-2\ell}\right).$$

The Levi subgroup $M_\ell^{(m)}$ associated to $N_\ell^{(m)}$ is isomorphic to $(G_1)^\ell \times G_{m-2\ell} \times (G_1)^\ell$, and the stabilizer of $\psi_\ell^{(m)}$ in $M_\ell^{(m)}$ under adjoint action is isomorphic to $G_{m-2\ell-1}$. We embed $G_{m-2\ell-1}$ into G_m via $g \mapsto \begin{pmatrix} g & \\ & I_{2\ell-1} \end{pmatrix}$, and let $R_\ell^{(m)} = N_\ell^{(m)} \cdot G_{m-2\ell-1}$. For irreducible admissible representations π and τ of G_m and $G_{m-2\ell-1}$ respectively, we call

$$\mathrm{Hom}_{R_\ell^{(m)}}\left(\pi, \tau^\vee \otimes \psi_\ell^{(m)^{-1}}\right)$$

the *local Bessel model* with respect to π and τ. Here $\tau^\vee$ denotes the smooth dual of τ. We note here that such spaces are basic objects considered in the local Gan-Gross-Prasad conjectures (see [3]).

Let $\chi = \chi_1 \otimes \cdots \otimes \chi_m$ be a unramified character of $T_m \simeq (\mathrm{GL}_1)^m$, and

$$I(\chi) = \left\{ f \in C^\infty(G_m) \mid f(tug) = \delta_m^{\frac{1}{2}}(t)\chi(t)f(g)\ \forall u \in U_m, t \in T_m, g \in G_m \right\}$$

be the unramified representation of G_m induced from χ. We denote by Φ_{K_m} the K_m-invariant function (spherical function) in $I(\chi)$ normalized to be $\Phi_{K_m} = 1$. Let $\mathcal{H}_m = C_c^\infty(K_m \backslash G_m / K_m)$ be the spherical Hecke algebra of G_m. One defines the Hecke eigenfunction by

$$\omega_\chi(\phi) = \int_G \Phi_{K_m}(g)\phi(g)\,\mathrm{d}g. \quad (\phi \in \mathcal{H}_m)$$

Similarly, for a unramified character $\eta = \eta_1 \otimes \cdots \otimes \eta_{m-2\ell-1}$ of $T_{m-2\ell-1}$, one also has unramified induced representation $I(\eta)$ of $G_{m-2\ell-1}$, and the Hecke eigenfunction ω_η on the Hecke algebra $\mathcal{H}_{m-2\ell-1}$. Then a *Whittaker-Shintani function* in this case is a function $F(g) \in C^\infty(G_m)$ satisfying

$$L(uk)R(k')F(g) = \psi_\ell^{(m)}(u)F(g), \quad \forall u \in N_\ell^{(m)}, k \in K_{m-2\ell-1}, k' \in K_m; \qquad (1)$$

$$L(\varphi)R(\phi)F(g) = \omega_\eta(\varphi)\omega_\chi(\phi)F(g), \quad \forall \varphi \in \mathcal{H}_{m-2\ell-1}, \phi \in \mathcal{H}_m. \qquad (2)$$

Here L and R in (1) are the left and right regular representations of G_m, and $L(\varphi)$ and $R(\phi)$ in (2) are left and right regular integration operators given by convolutions. We denote by $\mathrm{WS}(\chi, \eta)$ the space of Whittaker-Shintani functions with respect to χ and η. One has

$$\mathrm{Hom}_{R_\ell^{(m)}}\left(I(\chi), I(\eta)^\vee \otimes \psi_\ell^{(m)^{-1}}\right) \simeq \mathrm{WS}(\chi, \eta),$$

i.e. the space of Whittaker-Shintani functions with respect to χ and η is isomorphic to the local Bessel model with respect to the unramified representations $I(\chi)$ and $I(\eta)$.

In this paper, by adopting the calculation in [15] for the orthogonal group case, the author gives an explicit formula for the normalized Whittaker-Shintani function in $\mathrm{WS}(\chi, \eta)$. The main result is the following theorem.

Theorem 1. *For any unramified characters χ and η as above, we have*

$$\dim_{\mathbb{C}} \mathrm{WS}(\chi, \eta) = 1.$$

Let $F_{\chi,\eta}$ be the normalized function in $\mathrm{WS}(\chi, \eta)$ via $F_{\chi,\eta}(e) = 1$, then we have

$$F_{\chi,\eta}(t'\mu w_m t) = \prod_{i=1}^{m-2\ell-1} (1 - q^{-i}) \cdot \sum_{\substack{w \in W(G_m) \\ w' \in W(G_{m-2\ell-1})}} \mathbf{c}_{\mathrm{WS}}(w\chi, w'\eta) \cdot ((ww_m\chi)(\varpi)\delta_m^{\frac{1}{2}}(t)) \cdot ((w'\eta)^{-1}(\varpi)\delta_{m-2\ell-1}^{\frac{1}{2}}(t')), \tag{3}$$

here $W(G_m)$ is the Weyl group of G_m, w_m is the longest Weyl element in $W(G_m)$, μ is the representative such that $R_\ell^{(m)}\mu w_m B_m$ is the unique open double coset in $R_\ell^{(m)} \backslash G_m / B_m$, q is the cardinality of the residue field of F, ϖ is a uniformizer in F, and $\mathbf{c}_{\mathrm{WS}}(\chi, \eta)$ is the factor given by

$$\mathbf{c}_{\mathrm{WS}}(\chi, \eta) = \frac{\prod_{1 \le i \le m-2\ell-1, 1 \le j \le m} (1 - q^{-\frac{1}{2}}((\eta_i\chi_j)(\varpi))^{a_{i,j}})}{\prod_{1 \le i < j \le m} (1 - (\chi_i\chi_j^{-1})(\varpi)) \prod_{1 \le i < j \le m-2\ell-1} (1 - (\eta_i\eta_j^{-1})(\varpi))},$$

where $a_{i,j} = 1$ if $i + j \le m - \ell$ and $a_{i,j} = -1$ if $i + j > m - \ell$.

We remark here that one already knows that

$$\dim_{\mathbb{C}} \mathrm{WS}(\chi, \eta) \le 1$$

from the uniqueness of local Bessel models for any irreducible admissible representations (see [1] and [3]). On the other hand, the formula (3) can be viewed as an analogue of Casselman-Shalika formula for spherical Whittaker functions (see [2]). As mentioned above, this formula, together with some related factors obtained during its calculation, are necessary in the unramified calculations in [12] and [13].

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