数学研究及评论
Mathematical Research with Reviews
Issue 2 (2019) Art.32

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Linear transformations preserving the strong q-log-convexity of polynomials
Electron. J. Combin. 22 (2015), no. 3, Paper 3.26, 11 pp.

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收稿日期：2019年11月20日

A nonnegative integer sequence $\{a_n\}_{n\geq 0}$ is said to be *log-convex* (resp. *log-concave*) if, for any $n \geq 1$, $a_{n-1}a_{n+1} \geq a_n^2$ (resp. $a_{n-1}a_{n+1} \leq a_n^2$). Log-concave and log-convex sequences arise often in combinatorics and other branches of mathematics, we refer the reader to [1, 7, 8] for relevant studies on log-concavity and [6, 9] for log-convexity.

A nonnegative integer sequence $\{a_n\}_{n\geq 0}$ is said to be *strongly log-convex* (resp. *strongly log-concave*) if, for $n \geq m \geq 1$, $a_{n+1}a_{m-1} \geq a_n a_m$ (resp. $a_{n+1}a_{m-1} \leq a_n a_m$). Clearly, a strongly log-convex (resp. strongly log-concave) sequence is also a log-convex (resp. log-concave) sequence.

A sequence of real polynomials $\{f_n(q)\}_{n\geq 0}$ is said to be *q-log-convex* if, for any $n \geq 1$, the difference $f_{n+1}(q)f_{n-1}(q) - f_n^2(q)$ has nonnegative coefficients as a polynomial in q. Similarly, we can define the q-log-concavity of polynomial sequences. It is easy to see that if the sequence $\{f_n(q)\}_{n\geq 0}$ is q-log-convex (resp. q-log-concave), then for each fixed nonnegative number q, the sequence $\{f_n(q)\}_{n\geq 0}$ is log-convex (resp. log-concave).

A sequence of real polynomials $\{f_n(q)\}_{n\geq 0}$ is called *strongly q-log-convex* if, for any $n \geq m \geq 1$, the difference $f_{n+1}(q)f_{m-1}(q) - f_n(q)f_m(q)$ has only nonnegative coefficients as a polynomial in q. The strongly q-log-concavity of polynomial sequences can be defined similarly. Clearly, the strongly q-log-convexity (resp. strongly q-log-concavity) of polynomial sequences implies the q-log-convexity (resp. q-log-concavity). In recent years, the q-log-concavity and q-log-convexity of polynomials have been extensively studied.

By providing a sufficient criterion, Liu and Wang [6] obtained several linear transformations which preserve q-log-convexity in 2007, and hence proved the q-log-convexity of several polynomials such as the Bell polynomials, the q-Schröder numbers, and the central q-Delannoy numbers. The motivation of this paper is to study linear transformations that preserve the strongly q-log-convexity. Explicitly, given a triangular array $\{a(n,k)\}_{0\leq k\leq n}$ of nonnegative real numbers and a polynomial sequence $\{f_n(q)\}_{n\geq 0}$, the authors consider the linear transformation $\mathcal{B}_n(q) = \sum_{k=0}^{n} a(n,k)f_k(q)$ ($n \geq 0$). The main contribution of this paper is to give a sufficient condition of $\{a(n,k)\}_{0\leq k\leq n}$ implying the strong q-log-convexity of $\mathcal{B}_n(q)$ if $\{f_n(q)\}_{n\geq 0}$ is strongly q-log-convex. The approach that the authors use to develop their sufficient condition is similar to that of Liu and Wang [6] and it shows that they have quite deep insight.

For convenience, let $a(n,k) = 0$ unless $0 \leq k \leq n$. Suppose $m \geq n$. For $0 \leq t \leq m+n$, define

$$a_k(m,n,t) = a(n-1,k)a(m+1,t-k) + a(m+1,k)a(n-1,t-k) - a(m,k)a(n,t-k) - a(n,k)a(m,t-k)$$

if $0 \leq k < \frac{t}{2}$, and

$$a_k(m,n,t) = a(n-1,k)a(m+1,k) - a(n,k)a(m,k)$$

if t is even and $k = \frac{t}{2}$. The sufficient condition is as follows.

Theorem 1. *Let $\mathcal{A}_n(q) = \sum_{k=0}^{n} a(n,k)q^k$. Suppose that the triangle $\{a(n,k)\}$ of nonnegative real numbers satisfies the following two conditions:*

(C1) *The sequence of polynomials $\{\mathcal{A}_n(q)\}_{n\geq 0}$ is strongly q-log-convex.*

(C2) *There exists an index $r = r(m,n,t)$ such that $a_k(m,n,t) \geq 0$ for $k \leq r$ and $a_k(m,n,t) < 0$ for $k > r$.*

If the sequence $\{f_k(q)\}_{k\geq 0}$ is strongly q-log-convex, then the polynomial sequence $\mathcal{B}_n(q)$ form a strongly q-log-convex sequence.

As applications of Theorem 1, the authors obtain some linear transformations (including Morgan-Voyce transformation, binomial transformation, Narayana transformation of two kinds) that preserve the strong q-log-convexity. Their results not only extend some known results, but also imply the strong q-log-convexity of some sequences of polynomials.

The authors prove that Morgan-Voyce transformation preserves the strong q-log-convexity.

Proposition 2. *If $\{f_k(q)\}_{k\geq 0}$ is strongly q-log-convex, then $g_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k} f_k(q)$ form a strongly q-log-convex sequence.*

The fact that $a(n,k) = \binom{n+k}{n-k}$ satisfies Condition (C2) in the proof of Proposition 2 is rather computationally complicated and it shows the authors' patience and good analytic skills.

As direct corollaries, we can obtain the strong q-log-convexity of the q-Schröder numbers $r_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k} C_k q^{n-k}$ (C_k denotes the kth Catalan number), the q-central Delannoy numbers $D_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k}\binom{2k}{k} q^{n-k}$, and Bessel polynomials $B_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k}\binom{2k}{k} k!(\frac{q}{2})^k$. The strong q-log-convexity of $r_n(q)$ and $D_n(q)$ have been proved by Zhu [9], and the strong q-log-convexity of $B_n(q)$ has been proved by Chen *et al.* [4], so Proposition 2 is an extension of these known results.

The authors also prove that binomial transformation preserves the strong q-log-convexity.

Proposition 3. *If $\{f_k(q)\}_{k\geq 0}$ is strongly q-log-convex, then $b_n(q) = \sum_{k=0}^{n} \binom{n}{k} f_k(q)$ form a strongly q-log-convex sequence.*

The Bell polynomial $B(n,q)$ is defined as

$$B(n,q) = \sum_{k=0}^{n} S(n,k) q^k,$$

where $S(n,k)$ is the Stirling number of the second kind. Note that $B(n+1,q) = \sum_{k=0}^{n} \binom{n}{k} B(n,q)$, by induction and Proposition 3 the authors give a new proof for the strong q-log-convexity of $B(n,q)$, which was proved earlier by Chen *et al.* [4] and Zhu [9, 10].

The Narayana polynomials of type B $W_n(q)$ are defined by $W_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 q^k$. Its q-log-convexity was conjectured by Liu and Wang [6] and proved by Chen *et al.* [2]. The strong q-log-convexity of $W_n(q)$ has been obtained by Zhu [9]. The authors extend this result as follows.

Proposition 4. *If $\{f_k(q)\}_{k\geq 0}$ is strongly q-log-convex, then $s_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 f_k(q)$ form a strongly q-log-convex sequence.*

From Proposition 4 the strong q-log-convexity of the hypergeometric polynomials $P_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} q^k$ follows immediately, which extends Dou and Ren's earlier result [5] stating that $P_n(q)$ form a log-convex sequence. We can also immediately obtain the strong q-log-convexity of the rook polynomials $S_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 k! q^k$.

The Narayana polynomials $N_n(q)$ are defined to be $N_n(q) = \sum_{k=0}^{n} N(n,k) q^k$, where the Narayana number $N(n,k)$ denote the number of Dyck paths of length $2n$ with exactly k peaks and $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. The strong q-log-convexity of $N_n(q)$ has been proved by Chen *et al.* [2] and Zhu [9]. The authors extend this result as follows.

Proposition 5. *If $\{f_k(q)\}_{k\geq 0}$ is strongly q-log-convex, then $\mathcal{N}_n(q) = \sum_{k=0}^{n} N(n,k) f_k(q)$ form a strongly q-log-convex sequence.*

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