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On p.p. structural matrix rings
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A ring is called a left *p.p.ring* if every principal left ideal is projective. This type of ring seems to have been initially introduced by Hattori [1] in 1960. They have gained much attention since then. Another type of ring is called left *semihereditary* if its every finitely generated left ideal is projective. Throughout, we denote by $\mathbb{M}_{m\times n}(R)$ the set of all $m\times n$ matrices over (the ring) R and let $\mathbb{M}_n(R) := \mathbb{M}_{n\times n}(R)$, also we let $\mathbb{T}_n(R)$ be the $n\times n$ upper triangular matrix ring over R. Let us first state the following two results.

Theorem 1. [2, Theorem 4] *Let R be a ring. The following conditions are equivalent:*

(1) R is regular.
(2) $\mathbb{T}_n(R)$ is a left p.p. ring for all $n \geq 1$.
(3) $\mathbb{T}_n(R)$ is a left p.p. ring for some $n \geq 2$.

Proposition 2. [3] *A ring R is right (left) semihereditary if and only if every $n\times n$ matrix over R has principal right (left) ideals projective.*

In the paper under review, the authors mention that if R is a regular ring, then all matrix rings and upper triangular matrix rings over R are $p.p.$ rings, and the converse holds too. It is interesting to observe that matrix rings and upper triangular matrix rings are two special cases of *structural matrix rings*. So it is natural to ask the question:

Question 3. *Is every structural matrix ring over a regular ring a left p.p. ring? If not, then which structural matrix rings over a regular ring are left* $p.p.$ *rings?*

To define the structural matrix rings, we briefly introduce some general setting for the sake of convenience. For a binary relation θ on $\{1, 2, \ldots, n\}$, we could define an $n \times n$ Boolean matrix $B = (b_{ij})$, where $b_{ij} = 1$ if $(i, j) \in \theta$ and $b_{ij} = 0$ if $(i, j) \notin \theta$. Clearly, for a given fixed binary relation θ on $\{1, 2, \ldots, n\}$, the corresponding Boolean matrix $B = (b_{ij})$ is uniquely defined. Then one can get an associated additively closed subset $\mathbb{M}_n(B, R)$ of $\mathbb{M}_n(R)$, where

$$\begin{aligned}\mathbb{M}_n(B, R) &= \{(a_{ij}) \in \mathbb{M}_n(R) : b_{ij} = 0 \Rightarrow a_{ij} = 0\} \\ &= \{(a_{ij}) \in \mathbb{M}_n(R) : (i, j) \notin \theta \Rightarrow a_{ij} = 0\}\,.\end{aligned}$$

A quasi-order Boolean matrix is a Boolean matrix $B = (b_{ij})$ which is reflexive (i.e., $b_{ii} = 1$ for all i) and transitive (i.e., $b_{ij} = 1 = b_{jk}$ implies $b_{ik} = 1$ for all i, j, k).

Definition 4. *For an* $n \times n$ *quasi-order Boolean matrix* B*, the subring* $\mathbb{M}_n(B, R)$ *of* $\mathbb{M}_n(R)$ *is called the structural matrix ring over* R *associated with* B*.*

An $n \times n$ quasi-order Boolean matrix B is blocked triangular if it is of the form

$$\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ 0 & B_{22} & \cdots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{kk} \end{pmatrix},$$

where for every $i \leq j$, B_{ij} is an $n_i \times n_j$ (Boolean) matrix with all its entries equal, and $n_1 + \cdots + n_k = n$. If every entry of B_{ij} is 1 for all $i \leq j$, then B is called complete blocked triangular.

If $A = (a_{ij})$ is an $n \times n$ matrix and $\sigma \in \mathrm{Sym}(n)$, let A^σ denote the $n \times n$ matrix whose $(\sigma(i), \sigma(j))-$ entry is a_{ij} for all $i, j \in \{1, 2, \ldots, n\}$. Given an $n \times n$ quasi-order Boolean matrix B and $\sigma \in \mathrm{Sym}(n)$, B^σ is also a quasi-order Boolean matrix, and $\mathbb{M}_n(B, R) \cong \mathbb{M}_n\left(B^\sigma, R\right)$ via the isomorphism $A \mapsto A^\sigma$. Moreover, there exists a permutation $\sigma \in \mathrm{Sym}(n)$ such that B^σ is blocked triangular. If B is blocked triangular, then a structural matrix ring $\mathbb{M}_n(B, R)$ is called blocked triangular.

In this paper, the authors provide a negative answer to question 3 and they completely determine the structural matrix rings over a regular ring which are left $p.p.$

rings. Moreover, a new family of left $p.p.$ rings is established. Precisely, the main result is stated as follows:

Theorem 5. *Let R be a regular ring and B be an $n \times n$ quasi-order Boolean matrix with a blocked triangular Boolean matrix B^σ, with $\sigma \in$Sym(n). Write*

$$\mathbb{M}_n(B^\sigma, \mathrm{R}) = \begin{pmatrix} \mathbb{M}_{n_1}(R) & \mathbb{M}_{n_1 \times n_2}(X_{12}) & \dots & \mathbb{M}_{n_1 \times n_k}(X_{1k}) \\ 0 & \mathbb{M}_{n_2}(R) & \dots & \mathbb{M}_{n_2 \times n_k}(X_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbb{M}_{n_k}(R) \end{pmatrix}$$

where $X_{ij} = 0$ or $X_{ij} = R$ for all i, j with $1 \leq i < j \leq k$.

(1) If $k \leq 3$, then $\mathbb{M}_n(B, R)$ is a left p.p. ring.

(2) If $k \geq 4$, then $\mathbb{M}_n(B, R)$ is not a left p.p. if and only if there exist $1 \leq l_1 < l_2 < l_3 < l_4 \leq k$ such that $X_{l_2 l_3} = 0, X_{l_1 l_2} = X_{l_1 l_3} = X_{l_2 l_4} = X_{l_3 l_4} = R$.

The authors also obtain the following corollary and conclude that theorem 1 is its immediate consequence.

Corollary 6. *Let R be a ring and T be a complete blocked triangular matrix ring over R with block size greater than or equal to 2. Then R is a regular ring if and only if T is a left $p.p.$ ring.*

Remark 7. *In this paper under review, the authors conclude that a structural matrix ring T over a regular ring R is not a left $p.p.$ ring if and only if T is isomorphic to a blocked triangular matrix ring*

$$\begin{pmatrix} \mathbb{M}_{n_1}(R) & \mathbb{M}_{n_1 \times n_2}(X_{12}) & \dots & \mathbb{M}_{n_1 \times n_k}(X_{1k}) \\ 0 & \mathbb{M}_{n_2}(R) & \dots & \mathbb{M}_{n_2 \times n_k}(X_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbb{M}_{n_k}(R) \end{pmatrix},$$

where $X_{ij} = 0$ or $X_{ij} = R$ for all i, j with $1 \leq i < j \leq k$, such that $X_{l_2 l_3} = 0$ and $X_{l_1 l_2} = X_{l_1 l_3} = X_{l_2 l_4} = X_{l_3 l_4} = R$ for some $1 \leqslant l_1 < l_2 < l_3 < l_4 \leqslant k$.

Remark 8. *Corollary 6 provides a much easier way to determine whether a ring is regular or not.*

In the proof of the main result in this paper, the authors make full use of the following beautiful theorem.

Theorem 9. [4] *Every structural matrix ring over R is isomorphic to a blocked triangular matrix ring over R. Precisely, if B is an $n \times n$ quasi-order Boolean matrix, then B^σ is blocked triangular some $\sigma \in \mathrm{Sym}(n)$ and so*

$$\mathbb{M}_n(B,R) \cong \mathbb{M}_n\left(B^\sigma,R\right) = \begin{pmatrix} \mathbb{M}_{n_1}(R) & \mathbb{M}_{n_1\times n_2}\left(X_{12}\right) & \dots & \mathbb{M}_{n_1\times n_k}\left(X_{1k}\right) \\ 0 & \mathbb{M}_{n_2}(R) & \dots & \mathbb{M}_{n_2\times n_k}\left(X_{2k}\right) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbb{M}_{n_k}(R) \end{pmatrix},$$

where $X_{ij} = 0$ or $X_{ij} = R$ for all i, j with $1 \leq i < j \leq k$.

In this case, it suffices to study the isomorphic class $\mathbb{M}_n\left(B^\sigma,R\right)$ of a general structural matrix ring $\mathbb{M}_n(B,R)$, which has a good shape in the form of blocks as described above. Therefore, the induction on the number of the blocks of $\mathbb{M}_n\left(B^\sigma,R\right)$ can be applied.

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