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Homological dimensions relative to preresolving subcategories
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Homological dimensions are important numerical characteristics of an object in a category with respect to a certain specified class of objects in this category. For example, the classical projective, injective and flat dimensions of modules are defined relative to the classes of projective, injective and flat modules, respectively [EJ2]. In Gorenstein homological algebra, when the classes of projective, injective and flat modules are replaced by that of Gorenstein projective, Gorenstein injective, and Gorenstein flat modules respectively, we get the notions of Gorenstein projective, Gorenstein injective, and Gorenstein flat dimensions [C, CFH, CI]. Different homological dimensions have been extensively studied by many researchers and are interesting area in homological algebra [CI, EJ1, H1, Hu, HuH, LHX].

In the paper under review, the author introduces relative preresolving subcategories and precoresolving subcategories of an abelian category and defines homological dimensions and codimensions relative to these subcategories, respectively. Some properties of these homological dimensions and codimensions are studied. The paper is well structured and written.

Assume that $\mathscr{A}$ is an abelian category and all subcategories of $\mathscr{A}$ are full and additive.

Definition 1. *([Hu]) Let $\mathscr{C}$ be a subcategory of $\mathscr{A}$ and $n \geq 0$. For an object A in $\mathscr{A}$, the $\mathscr{C}$-dimension (resp. $\mathscr{C}$-codimension), denoted by $\mathscr{C}\text{-}\dim A$ (resp. $\mathscr{C}\text{-}\operatorname{codim} A$), is defined as inf$\{n \geq 0 \mid$ there exists an exact sequence $0 \to C_n \to \cdots \to C_1 \to C_0 \to A \to 0$ (resp. $0 \to A \to C^0 \to C^1 \to \cdots \to C^n \to 0$) in $\mathscr{A}$ with all C_i (resp. C^i) objects in $\mathscr{C}\}$.*

Definition 2. *Let $\mathscr{C} \subseteq \mathscr{T}$ be subcategories of $\mathscr{A}$.*

(1) ([SSW]) $\mathscr{C}$ is called a generator (resp. cogenerator) for $\mathscr{T}$ if for any object T in $\mathscr{T}$, there exists an exact sequence $0 \to T^{'} \to C \to T \to 0$ (resp. $0 \to T \to C \to T^{'} \to 0$) in $\mathscr{T}$ with C an object in $\mathscr{C}$.

(2) Let $\mathscr{E}$ be a subcategory of $\mathscr{A}$. $\mathscr{C}$ is called an $\mathscr{E}$-proper generator (resp. $\mathscr{E}$-coproper cogenerator) for $\mathscr{T}$ if for any object T in $\mathscr{T}$, there exists a $\operatorname{Hom}_{\mathscr{A}}(\mathscr{E}, -)$ (resp. $\operatorname{Hom}_{\mathscr{A}}(-, \mathscr{E})$)-exact exact sequence $0 \to T^{'} \to C \to T \to 0$ (resp. $0 \to T \to C \to T^{'} \to 0$) in $\mathscr{A}$ such that C is an object in $\mathscr{C}$ and $T^{'}$ is an object in $\mathscr{T}$.

Definition 3. *Let $\mathscr{E}$ and $\mathscr{T}$ be subcategories of $\mathscr{A}$. Then $\mathscr{T}$ is called $\mathscr{E}$-preresolving in $\mathscr{A}$ if the following conditions are satisfied.*

(1) $\mathscr{T}$ admits an $\mathscr{E}$-proper generator.

(2) $\mathscr{T}$ is closed under $\mathscr{E}$-proper extensions, that is, for any $\operatorname{Hom}_{\mathscr{A}}(\mathscr{E}, -)$-exact exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathscr{A}$, if both A_1 and A_3 are objects in $\mathscr{T}$, then A_2 is also an object in $\mathscr{T}$.

An $\mathscr{E}$-preresolving subcategory $\mathscr{T}$ of $\mathscr{A}$ is called $\mathscr{E}$-resolving if the following condition is satisfied.

(3) $\mathscr{T}$ is closed under kernels of $\mathscr{E}$-proper epimorphisms, that is, for any $\operatorname{Hom}_{\mathscr{A}}(\mathscr{E}, -)$-exact exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathscr{A}$, if both A_2 and A_3 are objects in $\mathscr{T}$, then A_1 is also an object in $\mathscr{T}$.

We fix a subcategory $\mathscr{E}$ of $\mathscr{A}$ and an $\mathscr{E}$-preresolving subcategory $\mathscr{T}$ of $\mathscr{A}$ admitting an $\mathscr{E}$-proper generator $\mathscr{C}$. The author mainly shows the following results.

Proposition 4. *Let*

$$0 \to M \to T_1 \xrightarrow{f} T_0 \to A \to 0 \tag{4.1}$$

be an exact sequence in $\mathscr{A}$ with both T_0 and T_1 objects in $\mathscr{T}$. Then we have

(1) There exists an exact sequence:

$$0 \to M \to T \to C \to A \to 0 \tag{4.2}$$

in $\mathscr{A}$ with T an object in $\mathscr{T}$ and C an object in $\mathscr{C}$.

(2) If (4.1) is $\mathrm{Hom}_{\mathscr{A}}(X,-)$*-exact for some object* X *in* $\mathscr{A}$*, then so is (4.2).*

Theorem 5. *The following statements are equivalent for any object* A *in* $\mathscr{A}$ *and* $n \geq 0$.

(1) $\mathscr{T}\text{-}\dim A \leq n$.

(2) There exists an exact sequence:

$$0 \to K_n \to C_{n-1} \to C_{n-2} \to \cdots \to C_0 \to A \to 0$$

in $\mathscr{A}$ *with all objects* C_i *in* $\mathscr{C}$ *and* K_n *an object in* $\mathscr{T}$.

Let $\mathscr{C}$ be a subcategory of $\mathscr{A}$. We denote by $\mathscr{C}^{\perp} = \{A$ is an object in $\mathscr{A} \mid \mathrm{Ext}^i_{\mathscr{A}}(C,A) = 0$ for any object C in $\mathscr{C}$ and $i \geq 1\}$ and ${}^{\perp}\mathscr{C} = \{A$ is an object in $\mathscr{A} \mid \mathrm{Ext}^i_{\mathscr{A}}(A,C) = 0$ for any object C in $\mathscr{C}$ and $i \geq 1\}$.

Theorem 6. *Let* $\mathscr{T} \subseteq \mathscr{C}^{\perp} \bigcap {}^{\perp}\mathscr{C}$ *and* $\mathscr{C}$ *be closed under direct summands. Then for an object* A *in* $\mathscr{A}$*,* $\mathscr{T}\text{-}\dim A = \mathscr{C}\text{-}\dim A$ *if one of the following conditions is satisfied.*

(1) $\mathscr{C}\text{-}\dim A < \infty$*,* $\mathscr{E} = \mathscr{C}$ *and* $\mathscr{T}$ *is closed under kernels of* $\mathscr{C}$*-proper epimorphisms.*

(2) $\mathscr{C}\text{-}\dim A < \infty$*,* $\mathscr{E} = \mathscr{C}$ *and* $\mathscr{C}\text{-}\dim^{<\infty}$ *is closed under direct summands.*

(3) A *is an object in* $\mathscr{T}^{\perp}$ *and* $\mathscr{C}$ *is a cogenerator for* $\mathscr{T}$.

Finally, some known results are generalized. The author also proposes some questions and conjectures, which are closely related to the Auslander-Reiten conjecture and the strong Nakayama conjecture.

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