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Liu, Yu (刘裕); Nakaoka, Hiroyuki

Hearts of twin cotorsion pairs on extriangulated categories

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评论员：周潘岳 (湖南理工学院，岳阳)

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The cotorsion pairs were first introduced by Salce in [S], and it has been deeply studied in the representation theory during these years, especially in tilting theory and Cohen-Macaulay modules. Later, the cotorsion pair are also studied in triangulated categories [IY], in particular, Nakaoka introduced the notion of hearts of cotorsion pairs and showed that the hearts are abelian categories [N1]. This is a generalization of the hearts of t-structure in triangulated categories [BBD] and the quotient of triangulated categories by cluster tilting subcategories [KZ]. Moreover, he generalized these results to a more general setting called twin cotorsion pair [N2]. More precisely, he introduced the notion of hearts of (twin) cotorsion pairs on triangulated categories and showed that they have structures of (semi-) abelian categories. Liu [L] also studied a twin cotorsion pair on an exact category with enough projectives and injectives and introduce a notion of the heart. He showed that its heart is preabelian. Moreover he showed the heart of a single cotorsion pair is abelian. These results are analog of Nakaoka's results in triangulated categories.

Recently, the notion of extriangulated categories was introduced in [NP] as a simultaneous generalization of triangulated categories and exact categories.

Triangulated categories and exact categories are extriangulated categories, while there are some other examples of extriangulated categories which are neither triangulated nor exact, see [NP, ZZ]. Hence, many results hold on triangulated categories and exact categories can be unified in the same framework.

We assume that $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category.

Definition 1. A pair of cotorsion pairs $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ on $\mathcal{B}$ is called a *twin cotorsion pair* if it satisfies $\mathbb{E}(\mathcal{S}, \mathcal{V}) = 0$, or equivalently $\mathcal{S} \subseteq \mathcal{U}$.

Definition 2. For any twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$, put $\mathcal{W} = \mathcal{T} \cap \mathcal{U}$ and call it the *core* of $(\mathcal{U}, \mathcal{V})$. Define as follows.

(a) $\mathcal{B}^+ = \mathcal{C}one(\mathcal{V}, \mathcal{W})$. Namely, $\mathcal{B}^+$ is defined to be the full subcategory of $\mathcal{B}$, consisting of objects B which admits a conflation

$$V_B \rightarrowtail W_B \twoheadrightarrow B,$$

where $W_B \in \mathcal{W}$ and $V_B \in \mathcal{V}$. It can be easily shown that we have $\mathcal{T} \subseteq \mathcal{B}^+$.

(b) $\mathcal{B}^- = \mathcal{C}o\mathcal{C}one(\mathcal{W}, \mathcal{S})$. Namely, $\mathcal{B}^-$ is defined to be the full subcategory of $\mathcal{B}$, consisting of objects B which admits a conflation

$$B \rightarrowtail W^B \twoheadrightarrow S^B,$$

where $W^B \in \mathcal{W}$ and $S^B \in \mathcal{S}$. It can be easily shown that we have $\mathcal{U} \subseteq \mathcal{B}^-$.

Definition 3. Let $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a twin cotorsion pair on $\mathcal{B}$, and write the quotient of $\mathcal{B}$ by $\mathcal{W}$ as $\underline{\mathcal{B}} = \mathcal{B}/\mathcal{W}$. For any morphism $f \in \mathcal{B}(X, Y)$, we denote its image in $\underline{\mathcal{B}}(X, Y)$ by $\underline{f}$.

For any full additive subcategory $\mathcal{C}$ of $\mathcal{B}$ containing $\mathcal{W}$, similarly we put $\underline{\mathcal{C}} = \mathcal{C}/\mathcal{W}$. This is a full subcategory of $\underline{\mathcal{B}}$ consisting of the same objects as $\mathcal{C}$.

Put $\mathcal{H} = \mathcal{B}^+ \cap \mathcal{B}^-$. Since $\mathcal{H} \supseteq \mathcal{W}$, we obtain a full additive subcategory $\underline{\mathcal{H}} \subseteq \underline{\mathcal{B}}$, which we call the *heart* of the twin cotorsion pair.

In this article, the authors study the heart of a cotorsion pairs on an exact category and a triangulated category in a unified method, by means of the notion of an extriangulated category. They prove that the heart is abelian, and construct a cohomological functor to the heart. That is to say, they show the following.

Theorem 4. For any twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ on $\mathcal{B}$, its heart $\underline{\mathcal{H}}$ is semi-abelian.

Theorem 5. For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ on $\mathcal{B}$, its heart $\underline{\mathcal{H}}$ is an abelian category.

Theorem 6. For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ on $\mathcal{B}$, the associated functor $H \colon \mathcal{B} \to \underline{\mathcal{H}}$ is cohomological.

If $\mathcal{B}$ has enough projectives, the subcategory of projectives is denoted by $\mathcal{P} \subseteq \mathcal{B}$, the authors prove that H gives an equivalence between the heart and the category of coherent functors over the coheart modulo projectives.

If $\mathcal{B}$ has enough projectives and enough injectives. The authors define n-cluster tilting subcategory $\mathcal{M}$ in $\mathcal{B}$ and show how cotorsion pairs are induced from $\mathcal{M}$. They also show how an n-cluster tilting subcategory of an extriangulated category gives rise to a family of cotorsion pairs with equivalent hearts.

Finally, the authors give an example of n-cluster tilting subcategory in an extriangulated category which is neither triangulated nor exact.

This article is not only very interesting, but also nicely written. So all together its reading is highly recommended for anyone willing to learn about this subject.

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