数学研究及评论
Mathematical Research with Reviews
Issue 2 (2019) Art.29

Han, Guo-Niu (韩国牛)
Hankel continued fraction and its applications
Adv. Math. 303 (2016), 295–321.

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收稿日期：2019年11月13日

Given a sequence $\mathbf{a} = \{a_n\}_{n\geq 0}$, taking values in a fied $\mathbb{F}$, its *Hankel determinant* of order n is defined by

$$H_n(\mathbf{a}) := \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{vmatrix}.$$

The Hankel determinants play an important role in the study of the irrationality exponent of real numbers [2, 3]. Hankel determinants of special sequences have been studied in [1, 4, 5, 7]. However, for a given sequence, it is very difficult to obtain its explicit Hankel determinant. Using the Jacobi continued fraction expansion, Han [6] gave short proofs of some Hankel determinant formulas. But the existence of the Jacobi continued fraction requires that all Hankel determinants are nonzero.

In this paper, Han introduced the *Hankel continued fraction*, whose existence and uniqueness are guaranteed without any restrictions on the sequence. More generally, Han defined a super δ-fraction for each positive integer δ. Let $f(x) = a_0 + a_1x +$

$a_2x^2 + \cdots \in \mathbb{F}[[x]]$ be the generating sequence of $\mathbf{a}$. Then the *super δ-fraction* is a continued fraction of the following form

$$f(x) = \cfrac{v_0 x^{k_0}}{1 + u_1(x)x - \cfrac{v_1 x^{k_0+k_1+\delta}}{1 + u_2(x)x - \cfrac{v_2 x^{k_1+k_2+\delta}}{1 + u_3(x)x - \ddots}}}, \tag{1}$$

where $v_j \neq 0$ are constants, k_j are nonnegative integers and $u_j(x)$ are polynomials of degree less than or equal to $k_{j-1} + \delta - 2$. By convention, 0 is of degree -1.

When $\delta = 2$ and all $k_j = 0$, the super δ-fraction (1) is the traditional J-fraction. A super 2-fraction is called *Hankel continued fraction* (*H-fraction,* for short). Sometimes, we denote $H_n(\mathbf{a})$ by $H_n(f(x))$ if $f(x) = a_0 + a_1x + a_2x^2 + \cdots \in \mathbb{F}[[x]]$. Usually, $a_0 = 1$. The following theorem tells us that the Hankel determinants can also be evaluated by using the Hankel continued fraction.

Theorem 1. *(1) Let δ be a positive integer. Each super δ-fraction defines a power series, and conversely, for each power series $f(x)$, the super δ-fraction expansion of $f(x)$ exists and is unique.*

(2) Let $f(x)$ be a power series such that its H-fraction is given by (1) with $\delta = 2$. Then, all non-vanishing Hankel determinants of $f(x)$ are given by

$$H_{s_j}(f(x)) = (-1)^{\epsilon_j} v_0^{s_j} v_1^{s_j - s_1} v_2^{s_j - s_2} \cdots v_{j-1}^{s_j - s_{j-1}},$$

where $\epsilon_j = \sum_{i=0}^{j-1} k_i(k_i+1)/2$ and $s_j = k_0 + k_1 + \cdots + k_{j-1} + j$ for every $j \geq 1$.

In particular, if a power series $f(x)$ over a finite field satisfies a quadratic equation, then the Hankel continued fraction is ultimately periodic.

Theorem 2. *Let p be a prime number and $f(x) \in \mathbb{F}_p[[x]]$ be a power series satisfying the following quadratic equation*

$$A(x) + B(x)f(x) + C(x)f(x)^2 = 0 \tag{2}$$

where $A(x), B(x), C(x) \in \mathbb{F}_p[[x]]$ are three polynomials with one of the following conditions

(i) $B(0) = 1, C(0) = 0, C(x) \neq 0;$

(ii) $B(0) = 1, C(x) = 0;$

(iii) $B(0) = 1, C(0) \neq 0, A(0) = 0;$

(iv) $B(x) = 0, C(0) = 1, A(x) = -(a_k x^k)^2 + O(x^{2k+1})$ *for some* $k \in \mathbb{N}$ *and* $a_k \neq 0$ *when* $p \neq 2$.

Then, the Hankel continued fraction expansion of $f(x)$ *exists and is ultimately periodic. Also, the Hankel determinant sequence* $H(f(x)) = \{H_n(f(x))\}_{n\geq 0}$ *is ultimately periodic.*

It is well known that the simple continued fraction for a real number r is infinite and ultimately periodic if and only if r is a quadratic irrational number. Theorem 2 above can be viewed as a power series analog of Lagrange's theorem for real numbers. Moreover, the converse of the first part of Theorem 2 is an analog of Euler's theorem. That is the following theorem.

Theorem 3. *Let* δ *be a nonnegative integer and* $f(x)$ *a power series. If the super* δ*-continued fraction expansion of* $f(x)$ *is ultimately periodic, then* $f(x)$ *satisfies the quadratic equation (2).*

As a application, the periodicity of Hankel determinant sequences for several automatic sequences can be proved. These results can be used for studying irrationality exponents [3].

Theorem 4. *For each pair of positive integers* a, b*, let*

$$G_{a,b}(x) = \frac{1}{x^{2a}} \sum_{n=0}^{\infty} \frac{x^{2^{n+a}}}{1 - x^{2^{n+b}}} \in \mathbb{F}_2[[x]].$$

Then the Hankel determinant sequence $H(G_{a,b})$ *is ultimately periodic.*

When $(a, b) = (0, 0)$, Theorem 3 reproves Coon's theorem [4]. When $(a, b) = (2, 1), (2, 0), (1, 1)$, Theorem 3 reproves Han's result [6]. When $(a, b) = (0, 2)$, Theorem 3 reproves the result of Guo, Wu and Wen [5]. $G_{0,2}$ is usually called *regular paperfolding sequence*.

Proposition 5. *Let* $\mathbf{r} = \{r_n\}_{n\geq 0}$ *be the Rudin-Shapiro sequence defined by* $r_0 = 0$, $r_{2n} = r_n$, $r_{4n+1} = r_n$, $r_{4n+3} = 1 - r_{2n+1}$ $(n \geq 0)$. *Then the Hankel determinant sequence* $H(\mathbf{r})$ *is ultimately periodic.*

Let $\mathbf{a} = \{a_n\}_{n\geq 0}$ be the *Stern's sequence* defined by $a_0 = 1, a_1 = 1, a_{2n} = a_n, a_{2n+1} = a_n + a_{n+1}$ $(n \geq 1)$. Let $\mathbf{b} = \{b_n\}_{n\geq 0}$ be the *twisted version of Stern's sequence* defined by $b_0 = 0, b_1 = 1, b_{2n} = -b_n, b_{2n+1} = -(b_n + b_{n+1})$ $(n \geq 1)$. Then, the following proposition holds.

Proposition 6. *Let* $A(x) = \sum_{n=0}^{\infty} a_{n+1}x^n$ *and* $B(x) = \sum_{n=0}^{\infty} b_{n+1}x^n$. *Then*

$$H_n(A(x))/2^{n-2} \equiv H_n(B(x))/2^{n-2} \equiv (0,0,1,1)^* (\mathrm{mod}\ 2)$$

In this paper, many examples and conjectures are listed at the end.

The main work of this paper is a discovery of the Hankel continued fraction whose existence is guaranteed without any restrictions on the sequence. It is a powerful tool to determine the Hankel determinants. This is the innovation of the paper. By H-fraction, the paper gives a description of power series over a finite filed that satisfies the quadratic equation, which is analogous to Lagrange's theorem for real number. This is the highlight of the paper. At last, this paper gives a sufficient condition to describe the periodicity of the Hankel determinant sequences. This is an interesting result.

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