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From simple-minded collections to silting objects via Koszul duality

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The paper under review (cited as [**SY19**] in the sequel) studies silting objects in derived categories. The notion of a silting object was introduced by Keller-Vossieck [1] as a generalization of a tiling object. Given a triangulated category $\mathscr{C}$ with direct sums, and denoted by $[1]$ the functor which shifts the degree down by 1, a *silting object* is an object X of $\mathscr{C}$ such that

- $\hom_{\mathscr{C}}(X, X[-p]) = 0$ for any $p > 0$;
- X split-generates $\mathscr{C}$.

Closely related to the notion of a silting object is a simple minded-collection. A set of objects $\{Y_1, \cdots, Y_r\}$ of $\mathscr{C}$ is *simple-minded* if

- $\hom_{\mathscr{C}}(Y_i, Y_j[p]) = 0$ if $p > 0$;
- $\hom_{\mathscr{C}}(Y_i, Y_j) = 0$ if $i \neq j$ and $End_{\mathscr{C}}(Y_i)$ is a division $\Bbbk$-algebra;
- $Y_1, \cdots, Y_r$ split-generate $\mathscr{C}$.

A simple minded collection $\{Y_1, \cdots, Y_r\}$ is called *elementary* if $End_{\mathscr{C}}(Y_i) \cong \Bbbk$ for all $1 \leq i \leq r$.

Let $\mathscr{A}$ be a dg (i.e., differential graded) algebra over $\Bbbk$, there are two associated triangulated categories: the derived category $D^{perf}(\mathscr{A})$ of perfect dg modules over $\mathscr{A}$, and the derived category $D^{prop}(\mathscr{A})$ of proper dg modules over $\mathscr{A}$. Suppose $\mathscr{A}$ is supported in non-positive degrees and locally finite (i.e. $H^*(\mathscr{A})$ finite dimensional in each fixed degree), denoted by $\mathscr{A}^! := R\hom_{\mathscr{A}}(\Bbbk, \Bbbk)$ the Koszul dual of $\mathscr{A}$, there is an equivalence

$$D^{perf}(\mathscr{A}) \cong D^{prop}\left(\mathscr{A}^!\right) \tag{1}$$

between triangulated categories; see for example Sections 2.4 and 4.1 of [3] for an explanation of this fact. In [**SY19**], the authors work under the more restrictive assumption that $\mathscr{A}$ is actually proper (i.e. the total dimension of $H^*(\mathscr{A})$ is finite), so $\mathscr{A}$ is actually self-Koszul dual, namely $\mathscr{A}^!$ is quasi-isomorphic to $\mathscr{A}$. The main result of [**SY19**] proves that under the equivalence (1), an elementary simple-minded collection in $D^{prop}(\mathscr{A})$ corresponds to a silting object in $D^{perf}(\mathscr{A})$. More precisely, the following statement is proved:

Theorem 1. **[SY19]** *Let $\mathscr{A}$ be a proper dg algebra supported non-positive degrees. Then for any elementary simple-minded collection $\{Y_1, \cdots, Y_r\}$ in $D^{prop}(\mathscr{A})$, there exists a unique (up to isomorphism) silting object $X = X_1 \oplus \cdots \oplus X_r$ of $D^{perf}(\mathscr{A})$ such that for $1 \leq i, j \leq r$, we have*

$$\hom_{D^{perf}(\mathscr{A})}(X_i, Y_j[-p]) = \begin{cases} \Bbbk & \textit{if } i = j \textit{ and } p = 0; \\ 0 & \textit{otherwise}. \end{cases}$$

A fact which is worthy to mention here is that if the dg algebra $\mathscr{A}$ is Koszul in the usual sense (so in particular $\mathscr{A}$ is formal), then a silting object in $D^{prop}(\mathscr{A})$ is actually a tilting object. This fact is proved in Lemma 4.3 of [**SY19**].

Theorem 1 has a geometric interpretation if one takes $\mathscr{A}$ to be the endomorphism algebra of a basis of Lefschetz thimbles in the Fukaya-Seidel category $\mathscr{F}(\pi)$ associated to some Lefschetz fibration $\pi : M \to \mathbb{C}$. In this case, the silting object and the elementary simple-minded collection in the theorem above correspond respectively to a basis of Lefschetz thimbles $\{\Delta_1, \cdots, \Delta_r\}$, and its dual basis $\left\{\Delta_1^!, \cdots, \Delta_r^!\right\}$. To define the latter, one takes the original vanishing paths $\gamma_1, \cdots, \gamma_r \subset \mathbb{C}$ defining $\Delta_1, \cdots, \Delta_r$, and rotate them by an angle of π to get a dual set of vanishing paths $\gamma_1^!, \cdots, \gamma_r^! \subset \mathbb{C}$, the dual thimbles $\Delta_1^!, \cdots, \Delta_r^!$ are Lagrangian balls in M obtained by parallel transport of the vanishing cycles of π along $\gamma_1^!, \cdots, \gamma_r^!$. For details, see the monograph [4]. More generally, one could consider partially wrapped Fukaya categories [5] to get more interesting geometric examples of the dg algebra $\mathscr{A}$.

As a corollary of Theorem 1, the authors also obtained the following:

Theorem 2. **[SY19]** *Let $\mathscr{A}$ be a dg algebra as in Theorem 1, but now over an algebraically closed field $\Bbbk$. There are one-to-one correspondences which commute with mutations and which preserve partial orders between*

- *equivalence classes of silting objects in $D^{perf}(\mathscr{A})$;*
- *isomorphism classes of simple-minded collections in $D^{prop}(\mathscr{A})$;*
- *bounded t-structures of $D^{prop}(\mathscr{A})$ with length heart;*
- *bounded co-t-structures of $D^{perf}(\mathscr{A})$.*

The results of this paper are interesting. However, from the point of view of Koszul duality, it would be more interesting to investigate the general case when $\mathscr{A}$ is a non-positively graded, homologically smooth dg algebra over $\Bbbk$ which is locally finite instead of finite-dimensional, in which case the Koszul dual $\mathscr{A}^!$ of $\mathscr{A}$ is still proper. This is the case, for example, when $\mathscr{A}$ is the Ginzburg dg algebra associated to certain quivers with potentials (Q, w), whose Koszul dual is a proper and cyclic A_∞-algebra introduced by Kontsevich-Soibelman [2], and they also arise naturally in geometric context as Chekanov-Eliashberg dg algebras of certain Legendrian submanifolds, or equivalently, the endomorphism A_∞-algebra of the wrapped Fukaya categories of certain Weinstein manifolds, some cool examples are found in [3]. As mentioned in the paper under review, an analogue of Theorem 2 for this more general case is the work in preparation of Keller and Nicolás. It is natural to expect that such a result, once suitably interpreted, would have some interesting geometric implications.

References

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