数学研究及评论
Mathematical Research with Reviews
Issue 2 (2019) Art.26

Zhang, Qi S. (张旗); Zhu, Meng (朱萌)

Li-Yau gradient bounds on compact manifolds under nearly optimal curvature conditions

J. Funct. Anal. 275 (2018), no. 2, 478–515.

评论员：王凤雨 (天津大学应用数学中心，天津)
收稿日期：2019年11月9日

By using integrability conditions on the negative part of the Ricci curvature, the famous Li-Yau gradient estimate for positive solutions to the heat equation is established on an n-dimensional compact Riemannian manifold M. According to Li-Yau [1], if the Ricci curvature Ric is bounded below by a constant $-K$, where $K \geq 0$, then any positive solution to the heat equation

$$\partial_t u = \Delta u$$

satisfies the gradient estimate

$$|\nabla \log u|^2 - \alpha \partial_t \log u \leq \frac{n\alpha^2 K}{2(\alpha - 1)} + \frac{n\alpha^2}{2t}, \quad \alpha > 1, t > 0.$$

When $K = 0$ one may let $\alpha \downarrow 1$ in this inequality to derive the sharp estimate $|\nabla \log u|^2 \leq \partial_t \log u + \frac{n}{2t}$.

Due to its importance in applications, this type inequality has been intensively extended and refined in various different situations. This paper aims to establish this inequality using integrability conditions of the negative part Ric^- of the Ricci curvature instead of the curvature lower bound, which is in particular interesting when singularities appear in the manifold. The heat equation for both a fixed metric and the

Ricci flow have been considered, where in the first case the inequality is derived under the condition $|\text{Ric}^-| \in L^p(dx)$ for some $p > \frac{n}{2}$, or $|\text{Ric}^-|^2 d(x, \cdot)^{2-n} \in L^1(dx)$ and the Gaussian upper bound of the heat kernel; while in the second case only the boundedness of the scale curvature is used. Remarks on the sharpness and the background of these conditions are presented.

The main results are new, and the method developed in the paper by constructing auxiliary function in calculations to cancel the bad term should be useful for further study. However, since the Ricci curvature is bounded for a smooth compact Riemannian manifold, it would be helpful to extend the main results to non-compact manifolds. Indeed, it is explained in the paper that in the second case the result can be extended also to non-compact manifolds. To cover the classical case that Ric is bounded from below, in the non-compact case it would be reasonable to adopt integrability conditions on Ric^- with respect to a probability measure instead of the volume measure.

REFERENCES

[1] Peter Li, Shing-Tung Yau, On the parabolic kernel of the Schrödinger operator, Acta Math. 156 (3–4) (1986) 153–201.