数学研究及评论
Mathematical Research with Reviews
Issue 2 (2019) Art.24

Fei, Jie (费杰); Wang, Jun (王军)
Rigidity of holomorphic curves in a hyperquadric Q_4
Differential Geom. Appl. 65 (2019), 78–92.

评论员：何玲 (天津大学应用数学中心，天津)
收稿日期：2019年11月5日

In the geometry of submanifolds, one of the fundamental problems is to study rigidity and homogeneity of submanifolds in a homogeneous space. The rigidity results on holomorphic curves and minimal surfaces in the complex projective space $\mathbb{C}P^n$ are well studied. Calabi [2] proved that a linearly full holomorphic curve in $\mathbb{C}P^n$ is completely determined by its first fundamental form, up to a rigid motion of $\mathbb{C}P^n$. Bolton, Jensen, Rigoli and Woodward [1] proved that two conformal minimal immersions from 2-sphere S^2 into $\mathbb{C}P^n$ are congruent if they have the same Kähler angles and induced metrics. By using Calabi's rigidity theorem, they completely classified all minimal two-spheres in $\mathbb{C}P^n$ with constant curvature, which are all homogeneous and well known as the Veronese sequences.

For the complex Grassmannians $G(k, n, \mathbb{C})$, Griffiths [4] studied the rigidity of holomorphic curves in $G(n, 2n, \mathbb{C})$ and proved that two non-degenerate holomorphic curves in $G(2, 4, \mathbb{C})$ are congruent if they have the same first and second fundamental forms. Chi and Zheng [3] classified all holomorphic curves from two-spheres into $G(2, 4, \mathbb{C})$ with constant curvature 2 into two classes, and they are not congruent in $G(2, 4, \mathbb{C})$. Recently, Fei and Xu [6] generalized the Griffiths' rigidity result to the case of non-degenerate holomorphic curves in $G(2, 6, \mathbb{C})$.

A complex hyperquadric Q_n is an another important symmetric space, which is a complex hypersurface of $\mathbb{C}P^{n+1}$ and defined by

$$Q_n = \{[Z = (z_1, \cdots, z_{n+2})] \in \mathbb{C}P^{n+1} |\, z_1^2 + \cdots + z_{n+2}^2 = 0\},$$

where $[Z = (z_1, z_2, \cdots, z_{n+2})]$ is the homogeneous coordinates of $\mathbb{C}P^{n+1}$. Q_n has a natural Kähler metric inherited from the Fubini-Study metric of $\mathbb{C}P^{n+1}$. But it doesn't have constant holomorphic sectional curvature, and therefore its geometry structure is much more sophisticated than that of $\mathbb{C}P^{n+1}$. In recent years, the geometry of minimal surfaces in Q_n has been widely studied (cf. [7], [8], [9], [10], [11], [12], [13]).

In the present paper under review, the authors study rigidity of a holomorphic curve in a hyperquadric Q_4 by the theory of Lie group and the method of moving frames which provide a powerful tool in studying rigidity problems (cf. [4]). The authors firstly obtained Gauss equation and Codazzi equations of a holomorphic curve in a hyperquadric Q_n, and also computed the Laplace of the square of the length of the second fundamental form. These equations are fundamental in studying the geometry of holomorphic curves in Q_n.

Theorem 1. *Let $f : M \to Q_n$ be a holomorphic immersion from a surface M into Q_n. Let $\|B\|^2$ be the square of the length of the second fundamental form of f. Then the Gauss equation of f is*

$$K = 4 - 2\tau^2 - \frac{1}{2}\|B\|^2,$$

where K is the Gaussian curvature with respect to the induced metric, and τ is a globally defined function of analytic type that values in $[0, 1]$.

Theorem 2. *Let $f : M \to Q_n$ be a holomorphic immersion. The Codazzi equations of f are*

$$da_\alpha - \mathtt{i}(\omega_{12} - 2\rho)a_\alpha + \sum_\beta \omega_{\alpha\beta} a_\beta = a_{\alpha,1}\varphi + \Big[(\frac{K}{2} - 1)X_\alpha - \sum_\beta S_{\alpha\beta} X_\beta\Big]\overline{\varphi},$$

where a_α are complex second fundamental forms, ρ, φ are complex-valued 1-forms, $\omega_{12},\ \omega_{\alpha\beta}$ are real-valued 1-forms and $a_{\alpha,1},\ X_\alpha,\ S_{\alpha\beta}$ are locally complex-valued smooth functions.

Theorem 3. *Let $f : M \to Q_n$ be a holomorphic immersion. Let a be a complex vector-valued function that denotes the second fundamental forms of f. Then*

$$\frac{1}{4}\Delta|a|^2 = |a|^2(2K - 5) - (K - 3)(1 - \tau^2) + (\frac{K}{2} - 1)^2 + 5|X \cdot a|^2 + |a_{,1}|^2,$$

where $X,\ a_{,1}$ are complex vector-valued functions and $\cdot$ represents the symmetry inner product.

Then as an application of these fundamental equations, the authors studied rigidity of holomorphic curves in Q_4 and made use of similar arguments as in [6] to prove that

Theorem 4. *Let $f : M \to Q_4$ be a linearly full holomorphic curve. Then, up to a rigidity, f is uniquely determined by its first and second fundamental forms.*

By using the results in [8], the authors determined a family of linearly full homogeneous holomorphic curves parameterized by a constant $\zeta \in (0, \frac{\pi}{2}]$, $f : M \to Q_4$ with constant curvature 2 and different second fundamental forms, defined by

$$z \mapsto [\cos\frac{\zeta}{2}(2z, 1-z^2, \mathtt{i}(1+z^2)), \mathtt{i}\sin\frac{\zeta}{2}(2z, 1-z^2, \mathtt{i}(1+z^2))],$$

where z is a local holomorphic coordinate of M. The result shows that the rigidity of holomorphic curves in Q_n does not hold if we only assume that they have the same first fundamental form. Among others, the previous fact indicates that it is usually difficult to classify holomorphic curves with constant curvature in Q_n.

He, Jiao and Zhou [5] studied the rigidity of holomorphic curves with constant curvature in $G(2,5,\mathbb{C})$, and determined two classes of linearly full non-homogeneous holomorphic curves from two-sphere into $G(2,5,\mathbb{C})$ with the same constant curvature $4/3$. Motivated by this result, we finally raise the following problem:

Problem 5. *Is it possible to construct some examples of non-homogeneous holomorphic curves from 2-sphere in Q_4, or classify holomorphic 2-sphere with constant curvature in Q_4?*

References

[1] J. Bolton, G.R. Jensen, M. Rigoli and L.M. Woodward, On conformal minimal immersions of S^2 into $\mathbb{C}P^n$, Math. Ann., 279(1988), 599-620.

[2] E. Calabi, Isometric embedding of complex manifolds, Ann. Math., 58(1953), 1-23.

[3] Q. Chi and Y. Zheng, Rigidity of pseudo-holomorphic curves of constant curvature in Grassmann manifold, Trans. Amer. Math. Soc., 313(1989), 393-406.

[4] P. Griffiths, On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry, Duke Math. J., 41(1974), 775-814.

[5] L. He, X.X. Jiao, X.Ch. Zhou, Rigidity of holomorphic curves of constant curvature in $G(2,5)$, Diff. Geom. Appl., 43(2015), 21-44.

[6] J. Fei and X.W. Xu, Local rigidity of holomorphic curves in the complex Grassmann manifold $G(2,6)$, J. Geom. Phys., 121(2017), 438-451.

[7] X.X. Jiao and J. Wang, Conformal Minimal Two-Spheres in Q_n, Sci. China Math., 54(4)(2011), 817-830.

[8] C.K. Peng, J. Wang and X.W. Xu, Minimal two-spheres with constant curvature in the complex hyperquadric, J. Math. Pures Appl., 106(2016), 453-476.

[9] J. Fei and J. Wang, Local Rigidity of Minimal Surfaces in A Hyperquadric Q_2, J. Geom. Phys., 133(2018), 17-25.

[10] X.X. Jiao, M.Y. Li and L. He, Classification of conformal minimal immersions of constant curvature from S^2 to Q_3, J. Math. Soc. Japan, 68(2)(2016), 863-883.

[11] X.X. Jiao and J. Wang, Minimal surfaces in a complex hyperquadric Q_2, Manu. Math., 140(2013), 597-611.

[12] J. Wang and X.W. Xu, Minimal surfaces in the complex hyperquadric Q_2 II, Proc. Amer. Math. Soc., 143(2015), 2693-2703.

[13] X. Zhong, J. Wang and X.X. Jiao, Totally real conformal minimal tori in the hyperquadric Q_2, Science China Math., 56(2013), 2015-2023.