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Sharp L^p decay of oscillatory integral operators with certain homogeneous polynomial phases in several variables
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Let $n \geq 2$. Consider the following oscillatory integral operator

$$T_\lambda f(x) := \int_{\mathbb{R}^n} e^{i\lambda S(x,y)} \psi(x,y) f(y)\, dy, \quad x \in \mathbb{R}^n,$$

where $\psi(x,y)$ is a smooth function supported in a compact neighborhood of the origin, $S(x,y)$ is a real-valued phase function on $\mathbb{R}^n \times \mathbb{R}^n$, and λ is a large parameter. For fixed λ, we know that T_λ defines a bounded linear operator from $L^2(\mathbb{R}^n)$ into itself.

Let $\|T_\lambda\|_p$ denote the norm of T_λ as an operator that maps $L^p(\mathbb{R}^n)$ into itself. A basic problem about this operator is determining the optimal rate of decay of the L^p operator norm $\|T_\lambda\|_p$ as λ tends to infinity. In one-dimensional case, Phong and Stein [3, 4, 5, 6] studied the sharp L^2 decay of oscillatory integral operators with phase functions varying from homogeneous polynomials to real-valued analytic functions. In their fundamental work, Phong and Stein also showed the relation between the decay rate and the Newton distance of $S(x,y)$. Later, the sharp L^2 estimate was extended to most C^∞ phases by Rychkov [7] and Greenblatt [1]. For the corresponding L^p decay estimates of T_λ, one can see Shi–Yan [8] and Xiao [9].

In higher dimensional case, Greenleaf et al.[2] established the L^2 decay estimates for oscillatory integral operators T_λ whose phase functions are generic homogeneous polynomials. In this paper, the authors obtain the L^p decay estimates in λ for oscillatory integral operators T_λ with certain homogeneous polynomial phases of degree d, that is,

$$S(x,y) := \sum_{|\alpha|+|\beta|=d} a_{\alpha,\beta} x^\alpha y^\beta.$$

Denoted by $S^d(\mathbb{R}^n \times \mathbb{R}^n)$ the space of all homogeneous polynomials of degree d on $\mathbb{R}^n \times \mathbb{R}^n$. We can assume that the phase function does not contain any monomial terms that are purely functions of x or of y, since these do not affect the operator norm, and we denote the space consisting of such polynomials by $\mathcal{O}^d(\mathbb{R}^n \times \mathbb{R}^n)$. In order to obtain the L^p decay estimates for oscillatory integral operators with homogeneous polynomial phases, we need to consider a family of analytic operators and use complex interpolation.

Definition 1. *A homogeneous phase function $S(x,y)$ is said to satisfy the rank one condition if*

$$\operatorname{rank}\big(S''_{xy}(x,y)\big) \geq 1, \quad \text{for all } (x,y) \neq (0,0),$$

i.e., if S''_{xy} has at least one nonzero entry at every point in $\mathbb{R}^n \times \mathbb{R}^n \backslash (0,0)$.

Definition 2. *The Hilbert–Schmidt norm of a matrix $A = (a_{ij})$ is defined by*

$$\|A\|_{HS} := \big[\operatorname{Tr}(A \cdot A^T)\big]^{1/2} = \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}.$$

Let $S(x,y)$ be a real-valued homogeneous polynomial in higher dimensions with degree d. For $z = \sigma + it \in \mathbb{C}$, the analytic families of operators are given by

$$T_\lambda^z f(x) := \int_{\mathbb{R}^n} e^{i\lambda S(x,y)} \|S''_{xy}\|_{HS}^z \psi(x,y) f(y)\, dy, \quad x \in \mathbb{R}^n.$$

In particular, $T_\lambda^0 = T_\lambda$. In this paper, the authors obtain the following results for T_λ^z.

Theorem 3. *Let $\sigma_1 = -n/(d-2)$ and $\sigma_2 = (d-2n)/2(d-2)$. If $S(x,y)$ satisfies the rank one condition, then the following L^2 decay estimates hold:*

$$\big\|T_\lambda^z\big\|_2 \lesssim \begin{cases} \lambda^{-1/2}, & \sigma > \sigma_2, \\ \lambda^{-1/2}\log\lambda, & \sigma = \sigma_2, \\ \lambda^{-[(d-2)\sigma+n]/d}, & \sigma_1 < \sigma < \sigma_2. \end{cases}$$

Here the notation $X \lesssim Y$ means that there exists a positive constant $C > 0$ such that $X \leq CY$.

Theorem 4. *Define the operator T_1^z as*

$$T_1^z f(x) := \int_{\mathbb{R}^n} e^{iS(x,y)} \|S''_{xy}\|_{HS}^{\sigma_1+it} \psi(x,y) f(y)\, dy, \quad x \in \mathbb{R}^n.$$

If $\|S''_{xy}\|_{HS}^{1/(d-2)}$ is a norm in $\mathbb{R}^n \times \mathbb{R}^n$, then T_1^z is bounded from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ with the operator norm less than $C(1+|t|)$, and $C > 0$ is a constant independent of the coefficients of $S(x,y)$.

The main result of this paper is stated as follows.

Theorem 5. *Suppose that $S(x,y) \in \mathcal{O}^d(\mathbb{R}^n \times \mathbb{R}^n)$ and $d > 2n \geq 4$. If $\|S''_{xy}\|_{HS}^{1/(d-2)}$ is a norm in $\mathbb{R}^n \times \mathbb{R}^n$, then we have*

$$\|T_\lambda\|_p \lesssim \begin{cases} \lambda^{-\delta/2}, & d/(d-n) < p < d/n, \\ \lambda^{-\delta/2}(\log\lambda)^\delta, & p = d/n \ \text{ or } \ p = d/(d-n), \\ \lambda^{-1/p'}, & 1 < p < d/(d-n), \\ \lambda^{-1/p}, & d/n < p < \infty, \end{cases}$$

where $\delta = \delta(S)$ is the Newton distance of $S(x,y)$. Moreover, the authors show that the decay is sharp when $d/(d-n) < p < d/n$.

The main theorem follows from the interpolation between L^2-L^2 decay estimate of T_λ^z and the H^1-L^1 boundedness property of T_λ^z.

The results obtained in this paper are interesting. It is recommended for both students and researchers working on harmonic analysis.

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