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Liang, Xin (梁鑫); Li, Ren-Cang (李仁仓)
*The hyperbolic quadratic eigenvalue problem*
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The paper by Liang and Li is a survey paper that summarizes fundamental mathematical characterizations of this type of matrix problems and optimization-based numerical algorithms for computing several extreme eigenvalues of these problems. The manuscript is similar to an encyclopedia in style on this topic, and can be used as a handbook chapter to search for a wide range of important properties of this type of problems. Since linear symmetric/Hermitian eigenvalue problems are ubiquitous in many science and engineering applications, and the hyperbolic quadratic eigenvalue problem is most similar to the former among all quadratic eigenvalue problems, this manuscript is potentially relevant to many readers with diverse background who are interested in the mathematical and algorithmic sides of the underlying problem.

The manuscript outlines both theoretical and algorithmic materials relevant to this topic. On the theoretical side, it summarizes the definitions, sufficient and/or necessary conditions of the hyperbolic eigenvalue problem, the ordering of the positive and negative type eigenvalues and the inertia of the corresponding matrix polynomial, its close or equivalent relations with a linearization form (matrix pencil), including their eigen-decomposition representations; more importantly, it collects the Courant-Fischer type and Wielandt-Lidskii type min-max principles of eigenvalues or a collection of eigenvalues, and Cauchy type interlacing theorems, all of which are fundamental for the development of optimization-based numerical algorithms for extreme eigenvalue computation. The perturbation analysis results align with the well-known conclusions for linear symmetric/Hermitian eigenvalue problems, showing that the eigenvalues are perfectly well-conditioned under structure-preserving perturbations.

On the numerical side, the manuscript explains why Rayleigh-Ritz projection produces the best approximation to the desired extreme eigenvalues from a given subspace of projection. It summarizes the basic (unpreconditioned) steep est descend/ascend methods, their variants with extended subspaces, and the methods of conjugate gradient, the preconditioned versions and block versions of these methods. It also provides a global and a local/asymptotic convergence analysis of the steepest descent method, and provides two perspectives to explain why preconditioning speeds up the convergence. Finally, numerical examples show the use and efficiency of some methods.

The manuscript is comprehensive (close to exhaustive) and thorough, and therefore could be an excellent 'root reference' pointing toward many relevant publications and results. The contribution from such a survey paper should be fully appreciated. The quality of the paper seems already have been improved from several referees, including a respected expert, several years ago. Here, I have a few suggestions for the authors to further improve the paper.

(1) There could be a short section including several (say, three) specific and widely-seen applications that yield a hyperbolic quadratic eigenvalue problem. This could attract more readers from science and engineering applications and have a potentially higher broader impact.
(2) If possible, the authors may consider making some clarifications in sections 3 and 4, what characterizations and conclusions are universal to more general quadratic eigenvalue problems, and what are unique to hyperbolic quadratic eigenvalue problems alone. I understand that some of the differentiation is difficult to organize and make clear, but some efforts and remarks to achieve this would be very worthwhile.
(3) Since the paper proposed the Wielandt-Lidskii type min-max principle for a collection of eigenvalues, the authors may also consider adding the trace minimization method ([1] by Sameh et al) to sections 8 to 11 for a brief discussion. At least, such an algorithm should be introduced and adapted to the setting of quadratic eigenvalue problems. Here, since the algorithm constructs and solves a Jacobi-Davidson-like equation for a correction direction, the authors

may consider how to define the corresponding equation with proper projectors. The authors may refer to reference [3] to consider this possibility.

(4) In recent years, algorithms for solving more general nonlinear eigenvalue problems $T(\lambda)v = 0$ that satisfy the Courant-Fischer type min-max principle and the Cauchy interlacing theorem have been discussed, for extreme and interior eigenvalues; see [2, 3]. It could be mentioned that the algorithms discussed in this paper have more general variants.

(5) The authors acknowledged that there is a gap between the significant amount of numerical evidence and the lack of theoretical understanding of the conjugate gradient methods for extreme eigenvalue computation. A pointer to a recent study on how the use of ‘search directions’ for conjugate gradient type methods asymptotically achieves ‘quasi global optimality’ for Rayleigh quotient minimization [4] could also be added. Though this reference did not give explicit rate of convergence, it showed that a locally optimized Rayleigh quotient would be increasingly close to a globally optimized Rayleigh quotient as the algorithm proceeds toward convergence, with numerical illustrations.

## References

[1] A. Sameh, Z. Tong, The trace minimization method for the symmetric generalized eigenvalue problem, J. Comput. Appl. Math., Vol. 123(1-2), 2000, pp. 155–175.

[2] D. B. Szyld, F. Xue, Preconditioned eigensolvers for large-scale nonlinear Hermitian eigenproblems with variational characterizations. I. Extreme eigenvalues, Math. Comp., Vol. 85(302), 2016, pp. 2887–2918.

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