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The degree of biholomorphic mappings between special domains in $\mathbb{C}^n$ preserving 0

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A classical theorem due to H. Cartan [1] asserts that any biholomorphic mapping between two bounded circular domains in $\mathbb{C}^n$ must be linear, provided that it preserves the origin. Later, this theorem was generalized to the biholomorphic mappings between quasi-circular domains by W. Kaup [3], which says that the biholomorphic mapping $f : \Omega_1 \to \Omega_2$ is a polynomial mapping, if Ω_1 and Ω_2 are invariant under the following diagonal S^1-action respectively:

$$(z_1, \cdots, z_n) \mapsto (\lambda^{p_1} z_1, \cdots, \lambda^{p_n} z_n),$$

where $|\lambda| = 1$ and p_i are positive integers. We say the mapping $f = (f_1, \cdots, f_n)$ is a polynomial mapping if each component f_k of f is a polynomial in z, for $k = 1, \cdots, n$.

After that, P. Heinzner [4] extended the S^1-action to a more general kind of action by an arbitrary compact Lie group. More precisely, let G be a compact Lie group and $\rho : G \to GL(\mathbb{C}^n)$ be a continuous representation. A domain Ω is called G-invariant if $\rho(g)\Omega = \Omega$ for any $g \in G$. Heinzner proved that any automorphism f of a G-invariant domain Ω must be a polynomial mapping, if all the G-invariant holomorphic functions are constant and $f(0) = 0$.

Recently, Ning-Zhang-Zhou [5] considered the biholomorphisms between two different G_i-invariant ($i = 1, 2$) domains Ω_1 and Ω_2. They obtained that the biholomorphism mapping between Ω_1 and Ω_2 is also polynomial, if the G_i-invariant holomorphic functions on Ω_i ($i = 1, 2$) are constant and the biholomorphism mapping preserves the origin.

In the paper under review, the authors continue to consider the estimate for the degree of the biholomorphism polynomial mapping between two G_i-invariant domains $f : \Omega_1 \to \Omega_2$, where G_i ($i = 1, 2$) are two arbitrary compact Lie groups. Recall that the degree of a polynomial mapping $f = (f_1, \cdots, f_n)$ is defined as $\deg f := \max_{k=1,\cdots,n} \deg f_k$. They proved that if the corresponding G_i-invariant holomorphic functions are constant and $f(0) = 0$, then the degree of the polynomial mapping f satisfies $\deg f \leqslant a(G_1)a(G_2)$, where $a(G_1), a(G_2)$ are two constants uniquely determined by the compact Lie groups G_1 and G_2 respectively (see [5, Theorem 3.4]).

When G_1 and G_2 are some special compact Lie groups, for instance the case of quasi-circular domains, the authors proved a more precise estimate for the degree of the polynomial biholomorphism between two quasi-circular domains, in terms of the defining parameters of the group G_i ($i = 1, 2$) (see [5, Theorem 2.2] for more details). Several previous estimates, for example, the degrees of automorphisms of quasi-circular domains [6], the degrees of biholomorphisms between bounded quasi-Reinhardt domains [2], i.e., $(S^1)^k$-invariant domains, can be derived from Theorem 2.2. Notice that, when the involved domain is bounded quasi-circular in $\mathbb{C}^2$, a complete classification of the automorphisms preserving the origin and a sharp estimate for the degree were given in [7]. However, when the dimension $n \geqslant 3$, a sharp estimate for the degrees of the biholomorphic polynomial mappings is unknown.

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