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Nonlinear Liouville problems in a quarter plane
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Cabré and Tan [1] studied the priori estimates of Gidas–Spruck type for

$$\begin{cases} A_{\frac{1}{2}}u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where where Ω is a smooth bounded domain of $\mathbb{R}^n$ and $A_{\frac{1}{2}}$ stands for the square root of the Laplacian operator in Ω with zero Dirichlet boundary values on $\partial\Omega$. For the precise definition of $A_{\frac{1}{2}}$, the readers are referred to the article of Cabré and Tan [1]. To prove this result, the method of blowing up plays an important role in the proof. In particular, two limiting equations when applying the method of blow-up to solutions of Equation (1) are obtained. One is

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ u > 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \dfrac{\partial u}{\partial \gamma} = u^p & \text{on } \partial\mathbb{R}^{n+1}_+, \end{cases} \tag{2}$$

It is well known that Equation (2) has no weak solutions for all $1 < p < \frac{n+1}{n-1}$ when $n \geq 2$. The other limiting equation is given by

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^{n+1}_{++}, \\ u > 0 & \text{in } \mathbb{R}^{n+1}_{++}, \\ u(0,y) = 0 & \text{on } \{x_n = 0, y \geq 0\}, \\ \dfrac{\partial u}{\partial \gamma} = u^p & \text{on } \{x_n > 0, y = 0\}, \end{cases} \tag{3}$$

where $\mathbb{R}^{n+1}_{++} = \{(x_1, x_2, \ldots, x_n, y) \in \mathbb{R}^{n+1} : x_n > 0, y > 0\}$. Cabré and Tan [1] obtained that for $n \geq 2$ and $p \leq \frac{n+1}{n-1}$, there exists no bounded classical solution to equation (3). In their proof, they combined the Kelvin transform and the method of moving planes to derive the symmetry of u with respect to x_i, $1 \leq i \leq n-1$. By the fact that equation (3) is translation invariant with respect to x_i, $1 \leq i \leq n-1$, it implies that u depends only on x_n and y. Hence, equation (3) is reduced to the following problem in the two-dimensional quarter plane

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^{2}_{++}, \\ u > 0 & \text{in } \mathbb{R}^{2}_{++}, \\ u(0,y) = 0 & \text{on } \{x_n = 0, y \geq 0\}, \\ \dfrac{\partial u}{\partial \gamma} = u^p & \text{on } \{x_n > 0, y = 0\}. \end{cases} \tag{4}$$

By the Hamiltonian identity for the half-Laplacian, they proved that Equation (4) has no bounded classical solution.

In this paper, Xiang removed their boundedness assumption since reducing equation (3) to (4) the boundedness assumption of the solution is not needed. And the boundedness assumption is only used when deriving the nonexistence of solutions of equation (4). In the proof of the result, as the first step, Xiang still uses the symmetry result of Cabré and Tan [1] and equation (3) is reduced to equation (4). Next, the author overcome the difficulties to obtain a monotonicity result, to be precise, they showed that any positive solution of equation (4) is monotone increasing in the x-direction, which is inspired by the work of Li and Lin [2] , where they employed the method of moving spheres. At last, together with the very general result of Cabré and Tan [1, Proposition 6.2], the author finishes the proof.

This paper is interesting, clear and straightforward, it uses the method of moving sphere to obtain monotonicity result, and then removes the condition of boundedness.

I strongly recommend that readers carefully study this method and use it to solve more related elliptic problems.

REFERENCES

[1] Cabré. X., and J. Tan. “Positive solutions of nonlinear problems involving the square root of the Laplacian.” Advances in Mathematics 224, no. 5 (2010): 2052–93.

[2] Li, Y. Y., and C.-S. Lin, “A nonlinear elliptic PDE and two Sobolev-Hardy critical exponents.” Archive for Rational Mechanics and Analysis 203, no. 3 (2012): 943–68.