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Generalized Choi states and 2-distillability of quantum states

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In quantum information theory, the maximally entangled pure states are the perfect resource in many quantum information processings [1]. Entanglement distillation is a process converting mixed entangled states into the maximally entangled state by using many copies of the entangled resource [2]. A quantum state or density matrix is a normalized positive semidefinite matrix mathematically. A quantum state ρ is positive under partial transpose (PPT) if $(id \otimes \tau)\rho$ is positive with id the identity map and τ the transpose map respectively, otherwise it is not positive under partial transpose (NPT). If quantum state ρ is PPT, then it is undistillable [3]. But whether all NPT states can be distilled is still a well known open problem. It is believed the answer is negative [4], but there is neither rigorous proof nor counterexample. In the paper under review, the authors investigated this open problem via the positive and completely positive maps.

First, the authors gave a necessary and sufficient condition for the distillability of the generalized Choi state. For the generalized Choi map $\Phi[a, b, c]$:

$M_3(\mathbb{C}) \to M_3(\mathbb{C})$ defined by [5, 6]

$$\Phi[a,b,c](X) = \begin{pmatrix} ax_{11}+bx_{22}+cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11}+ax_{22}+bx_{33} & -x_{23} \\ -x_{31} & -x_{32} & bx_{11}+cx_{22}+ax_{33} \end{pmatrix}$$

for $X = [x_{ij}] \in M_3(\mathbb{C})$, where a, b, c are nonnegative real numbers, the generalized Choi (GC) state is constructed as $(id_3 \otimes \Phi[a,b,c])\rho$. For maximally entangled state ρ, the GC state reduces to the Choi matrix [5]

$$A[a,b,c] := \begin{pmatrix} a & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & a & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & a \end{pmatrix}.$$

The authors proved that $A[a,b,c]$ is 1-distillable $\Leftrightarrow$ it is NPT $\Leftrightarrow$ $bc < 1$ by Peres-Horodecki criterion [7].

In fact all bipartite NPT state can be converted into $n \times n$ NPT Werner states $\rho(p,\ n) := I \otimes I - p\sum_{i,j=1}^{n} |ij\rangle\langle ji|$ for $p \in [-1,1]$, $n \geq 2$ [4]. When $n = 2$, all NPT state is trivially distillable. So the distillability problem is equivalent to the distillability of NPT Werner states with $n \geq 3$. $\rho(1/2, n)$ is called as the critical Werner state and it can be converted into $\rho(p,\ n)$ with $p \in (0, 1/2)$. The authors of the present paper considered the distillability of the critical Werner state and some valuable results are presented.

Employing the Werner map constructed as $\tau_n \circ \psi_n = Tr_n(\cdot)I_n - id_n/2$, where $\psi_n \in B(M_n(\mathbb{C}), M_n(\mathbb{C}))$ is a map as: $\psi_n(X) := \frac{1}{2}Diag(X) - \frac{1}{2}(X^\Gamma - Diag(X))$, X^Γ is the transpose of X and $Diag(X)$ is the matrix formed by the diagonal entries of X respectively, the authors proved the equivalence of the statements: the critical Werner state $\rho(1/2, n)$ is the 2-undistillable $\Leftrightarrow$ The map $(\tau_n \circ \psi_n)^{\otimes 2}$ is 2-positve when $n > 2$ $\Leftrightarrow$ $D(\rho_{ABC}) := \rho_A \otimes I_B \otimes I_C + \frac{1}{4}\rho_{ABC} - \frac{1}{2}\rho_{AB} \otimes I_C - \frac{1}{2}\rho_{AC} \otimes I_B$ is positive for all rank one density matrices ρ_{ABC}. The last statement makes the 2-distillability problem of critical Werner state more clear.

As a special case, the authors studied the positivity of $D(\rho_{ABC})$ with $n = 3$ by the classification of $2\times3\times3$ pure states in Ref. [8]. For this case, the positivity of $D(\rho_{ABC})$ for all rank one density matrices ρ_{ABC} is restricted to $\rho_{ABC} = (I_2 \otimes W \otimes X)|\Psi_1\rangle\langle\Psi_1|(I_2 \otimes W^\dagger \otimes X^\dagger)$, with $|\Psi_1\rangle := |000\rangle + |111\rangle + |022\rangle + |122\rangle$,

$$W = \begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & 0 & e \end{pmatrix}, \quad X = \begin{pmatrix} 1 & f & g \\ 0 & h & j \\ 0 & 0 & k \end{pmatrix}$$

up to a global factor, where $c, e, h, k > 0$. This reduces largely the number of parameters in the study of the positivity of $D(\rho_{ABC})$.

The distillability problem is rather difficult and there is no definite answer up to now. The reason lies in two facts. One is that the distillation requires many copies of quantum states and the other is that the quantum state itself is high dimensional. To solve this problem, one may begin with 2-distillability problem that requires only two copies. Since 2-distillability problem is also not easy and unsolved, one could consider the 2-distillability of $3\otimes3$ quantum states since the simplest case always gives us some clues for the general case. In this context, the present work promotes the research on the distillation problem and provides some nice results about 2-distillability.

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