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Some eigenvalue comparison theorems of Finsler p-Laplacian
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In geometric analysis, one of the fundamental problems is to study the eigenvalue comparison theorems for the Riemannian Laplacian, such as Cheng inequality, Cheeger inequality, Faber-Krahn inequality and McKean inequality.

Let M^n be an n-dimensional complete Riemannian manifold and $\Omega \subset M^n$ be a bounded domain with $\partial\Omega \in C^2$. Let $\lambda \in \mathbb{R}$ and consider the Dirichlet eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega. \end{cases} \tag{1}$$

It is well known that the spectrum of the Dirichlet problem (1) is discrete and satisfies the following:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \longrightarrow \infty,$$

where each eigenvalue is repeated according to its multiplicity.

Let us first state some eigenvalue comparison theorems.

Theorem 1. *(Cheng inequality; see [3]) Let M be a complete Riemannian n-manifold. If the Ricci curvature satisfies $Ric \geq (n-1)k$, then, for $x_0 \in M$ we* *have*

$$\lambda_1(B_{x_0}(r)) \leq \lambda_1(V_n(k,r))$$

and equality holds iff $B_{x_0}(r)$ is isometric to $V_n(k,r)$. Here $B_{x_0}(r)$ denotes the open geodesic ball with center x_0 and radius r, and $V_n(k,r)$ denotes a geodesic ball with radius r in the n-dimensional simply connected space form with sectional curvature k.

Theorem 2. *(Cheeger inequality; see [4]) Let M be a compact Riemannian manifold of dimension n. Define the Cheeger constant*

$$h = \inf_N \frac{area(N)}{\min(vol(A), vol(B))},$$

where N runs over (possibly disconnected) hypersurfaces of M which divides M into two disjoint open submanifolds A and B. Here $area(\cdot)$ denotes the $(n-1)$-dimensional volume, and $vol(\cdot)$ denotes the n-dimensional volume, respectively. Then

$$\lambda_1 \geq \frac{1}{4}h^2.$$

Theorem 3. *(Faber-Krahn inequality; see [5, 7]) For a bounded domain $\Omega \subset \mathbb{R}^n$, the first Dirichlet eigenvalue satisfies*

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^\star)$$

and equality holds iff $\Omega = \Omega^\star$. Here $\Omega^\star$ is a round ball in $\mathbb{R}^n$ with the same volume as Ω.

Theorem 4. *(McKean inequality; see [8]) Let M be a simply connected complete Riemannian n-manifold with sectional curvature $K \leq k < 0$. The spectrum of the Laplace operator acting in $L^2(M)$ (with the usual measure on M) lies to the left of $(n-1)^2k/4 < 0$.*

As generalizations, the above eigenvalue comparison theorems for the p-Laplacian were obtained by Matei [9] and Takeuchi [10]. The above eigenvalue comparison theorems are also valid for Finsler manifolds. We start by briefly introducing Finsler manifolds for the sake of convenience.

Let M be an n-dimensional manifold. A *Finsler metric* F on M means a function $F : TM \longrightarrow [0, \infty)$ with the following properties: (1) F is smooth on $TM\backslash\{0\}$; (2) $F(x, \lambda y) = \lambda F(x,y)$ for any $(x,y) \in TM$ and all $\lambda > 0$; (3) F is strongly convex, i.e., the matrix $[g_{ij}(x,y)] = [\frac{1}{2}(F^2)_{y^iy^j}]$ is positive definite for any nonzero $y \in T_xM$. Such a pair (M, F) is called a *Finsler manifold* and $g := g_{ij}(x, y)dx^i \otimes dx^j$ is called the *fundamental tensor* of F. We say that the *Finsler manifold* (M, F) is *Minkowskian*, *Riemannian* and *Euclidean* described as $g_{ij}(x, y) = g_{ij}(y)$, $g_{ij}(x, y) = g_{ij}(x)$ and $g_{ij}(x, y) = \delta_{ij}$, respectively. We define the *reverse metric* $\overleftarrow{F}$ of F by $\overleftarrow{F}(x, y) = F(x, -y)$ for all $(x, y) \in TM$, which is also a Finsler metric on M. A Finsler metric F on M is called *reversible* if $\overleftarrow{F}(x, y) = F(x, y)$ for all $(x, y) \in TM$.

Finsler metric is not reversible in general. This means that the distance function on a Finsler manifold may be not symmetric. On the other hand, the global analysis or topology on a Riemannian manifold is closely related to the theory of the Laplace operator (the elliptic linear operator), while the Finsler Laplacian is a fully nonlinear and degenerate operator on a whole Finsler manifold. Thus, the standard linear theory of PDEs cannot be directly applied to Finsler geometry. Despite this, the eigenvalue comparison theorems of the Finsler Laplacian was obtained by Ge-Shen [6], Chen [2], Wu-Xin [11] and Yin-He in [13, 12].

Now the p-Laplacian considered on a Finsler measure space (M, F, m) was discussed in [14], and is defined by

$$\Delta_{p,m} u := div_m[F^{p-2}(x, \nabla u)\nabla u], \qquad 1 < p < \infty,$$

where the gradient ∇ is a nonlinear operator and the equality holds in the weak $W^{1,p}(M)$ sense. It shows that any eigenfunction u (Dirichlet or Neumann) of Finsler p-Laplacian is $C^{1,\alpha}(M) \cap C^\infty(M_u \setminus M_0)$, where $M_u := \{du \neq 0\}$, $M_0 := \{u = 0\}$. In particular, when $p = 2$, $\Delta_{p,m}$ is exactly the usual Finsler Laplacian. If F is Riemannian, then the Finsler p-Laplacian is reduced to the corresponding one in the Riemannian geometry.

The first eigenvalue $\lambda_{1,p}$ of the Finsler p-Laplacian is the smallest positive number such that there exists a nonzero $u \in W^{1,p}(M)$ satisfying

$$\Delta_{p,m} u = -\lambda_{1,p}|u|^{p-2}u.$$

It has the following variational characterization for $\partial M = \emptyset$ or the Neumann problem

$$\lambda_{1,p} = \inf \left\{ \frac{\int_M [F^*(\nabla u)]^p \, dm}{\int_M |u|^p \, dm} \right| \; u \in W^{1,p}(M)\backslash\{0\}, \;\; \int_M |u|^{p-2}u \, dm = 0 \right\},$$

where F^* is the dual norm of F

It is natural to ask if one can extend the above eigenvalue comparison theorems for the p-Laplacian on a Finsler measure space (M, F, m) with certain curvature conditions.

In the paper under review, the authors extend the above eigenvalue comparison theorems for the p-Laplacian on a Finsler measure space (M, F, m), which generalize the corresponding theorems in Riemannian geometry and improve some results in Finsler geometry. More precisely, the author obtains the following comparison theorems.

Theorem 5. *(Cheng type inequality) Let (M, F, m) be a complete Finsler n-manifold. If the weighted Ricci curvature satisfies $Ric_N \geq (N-1)k$, $N \in [n, \infty)$, then the first Dirichlet eigenvalue of Finsler p-Laplacian satisfies*

$$\max\{\lambda_{1,p}(B^+_{x_0}(r)), \lambda_{1,p}(B^-_{x_0}(r))\} \leq \tilde{\lambda}_{1,p}(\overline{V}_{\overline{N}}(k, r)),$$

where $B^+_{x_0}(r))$ (respectively $B^-_{x_0}(r)$) denotes the forward (respectively backward) geodesic ball centered at x_0 of radius r, $\overline{V}_{\overline{N}}(k, r)$ denotes a (forward or backward) geodesic ball with radius r in the N-dim simply connected Finsler spaces with flag curvature k and vanishing S curvature, N is the smallest integer that is not less than N. Moreover, in all constant curvature spaces with vanishing S curvature and for both forward and backward geodesic balls, $\tilde{\lambda}_{1,p}$ are the same.

Theorem 6. *(Cheeger type inequality) Let (M, F, m) be a complete Finsler n-manifold. For any bounded domain Ω with piecewise smooth boundary in M, then the first Dirichlet eigenvalue of Finsler p-Laplacian satisfies*

$$\lambda_{1,p}(\Omega) \geq \left(\frac{h(\Omega)}{p}\right),$$

where $h(\Omega)$ is Cheeger constant.

Theorem 7. *(Faber-Krahn type inequality) Let $(\mathbb{R}^n, F, m)$ be a be a reversible Minkowski space. Let $\Omega \subset \mathbb{R}^n$ be a compact domain with smooth boundary and B be a metric ball in $(\mathbb{R}^n, F, m)$. If $vol^{dm}(\Omega) = vol^{dm}(B)$, then the first Dirichlet eigenvalue of Finsler p-Laplacian satisfies*

$$\lambda_{1,p}(\Omega) \geq \lambda_{1,p}(B),$$

where the equality holds if and only if Ω is isometric to B.

Theorem 8. *(McKean type inequality) Let (M, F, m) be a complete noncompact and simply connected Finsler n-manifold with nonpositive S curvature. If the flag curvature satisfies $K \leq -a^2 (a > 0)$, then*

$$\lambda_{1,p}(M) \geq \left(\frac{(n-1)a}{p}\right)^p.$$

Remark 9. Here, for flag curvature, weighted Ricci curvature and S curvature and so on, we refer to [1]. When $p = 2$, Chen [2] obtained the Theorem 6 but the manifold must have finite reversibility, and then the condition was removed by Yin-He [12]. When $p = 2$, the Theorem 7 was obtained by Ge-Shen in [6]. If the manifold has finite reversibility, the Theorem 8 was obtained by Wu-Xin [11] for $p = 2$ and by Yin-He [13] for general p. So these generalize the corresponding theorems in Riemannian geometry and improve some results in [2, 11, 12, 13].

The proof of the above theorems is similar to the Riemannian case, and the main steps are:
1. By using the first eigenfunction and distance function, we need to construct some test functions. With them, some desired inequalities are then obtained.
2. Since the gradient, p-Laplacian are nonlinear operators, we need to take care of the non-reversibility and give additional discussion when the eigenfunction changes its sign. In this case, some weighted operators are used if necessary.
3. Using Co-Area formula, Laplacian comparison theorem, divergence theorem and some properties of the operators, we complete the proofs.

Although the techniques and results in this paper seem somewhat parallel to the (unweighted) Riemannian case, the Finsler case needs to overcome some obstructions caused by the nonlinearity of p-Laplacian, lower order regularity for the eigenfunction and the non-reversibility of the Finsler metric as well as the effect of the weight N.

The reviewer thinks that these results are meaningful contributions to geometric analysis and Finsler manifolds. It is recommended reading for both students and advanced researchers.

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