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Canonical left cells and the lowest two-sided cell in an affine Weyl group

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Let W be a Coxeter group and let H be its Hecke algebra over an algebraically closed field k with deformation parameter q. The vector space H has a base (T_w) indexed by $w \in W$ coming from the the Matsumoto section $W \longrightarrow H$.

In their seminal work [2], Kazhdan and Lusztig introduced a second basis of H, called canonical basis $(C_w)_{w \in W}$, characterized by involuation-invariance and a triangularity property with respect to the Bruhat order $\leq$ on W. The change-of-basis matrix from (T_w) to (C_w) being upper triangular, its non-zero entries are polynomials $P_{y,w}(q)$ in q for $y \leq w$ (nowadays called Kazhdan–Lusztig polynomials). Considerations of maximal degrees of these polynomials lead to three partitions of W: into left cells, into right cells and into two-sided cells.

The cell structure of W is intimately linked to the representation theory of H. For example, if c is a left cell, then the sub-vector-space of H spanned by the C_w for $w \in c$ is a left ideal, called a cell module.

It is a challenging problem to describe the cells, as they involve the rather complicated polynomials $P_{y,w}(q)$. Since their introduction, there have been intensive works on the cell structure, mostly for finite Coxeter groups and for affine Weyl groups. Beyond these two cases, not much is known.

Now let us restrict to the affine case. A special feature in this case is that there exists a distinguished two-sided cell, the lowest two-sided cell c_0. Roughly speaking, c_0 contains most of the elements of W.

We recall several structural results of the affine Hecke algebra. The affine Weyl group is a semi-direct product $W = W_0 \ltimes P$, where W_0 and P are the Weyl group and the weight lattice of a finite-dimensional simple Lie algebra. Such a decomposition can be lifted: the affine Hecke algebra H contains two subalgebras, H^f the Hecke algebra of the finite Weyl group W_0, and $k[P]$ the group algebra of P; the multiplication map is an isomorphism of vector spaces

$$H^f \otimes k[P] \longrightarrow H, \quad x \otimes y \mapsto xy.$$

The center $Z(H)$ of H turns out to be the fixed point subalgebra $k[P]^{W_0}$. This implies that all central characters are restriction to $Z(H)$ of characters of P, which in turn can be identified with elements of a maximal torus of the Lie group. Let H_t be the quotient of H by the two-sided ideal generated by $z - t(z)$ for $z \in Z(H)$. Then H_t is of dimension $|W_0|^2$. Denote the quotient map by $\pi_t : H \longrightarrow H_t$.

Let w_0 be the longest element of W_0. The left ideal HC_{w_0} is indeed a cell module, and it appeared previously in [1] as an equivariant K-group of the cotangent bundle of a flag variety. In this paper, the author studies submodules of these cell modules by passing to the quotient $\pi_t : H \longrightarrow H_t$.

One of the main results of this paper is a realization of a family of irreducible H-modules. Let $\theta_\rho \in P \subset H$ be the sum of all fundamental weights. Let $(C'_w)_{w\in W}$ be a third basis of H obtained from the $P_{y,w}(q^{-1})$.

- If the element $\pi_t(C_{w_0}\theta_\rho C'_{w_0}) \in H_t$ does not vanish, then the left ideal of H_t generated by it is an irreducible H-module.

To make connections with cells, observe that: for a given irreducible H-module L, there exists a unique two-sided cell c of W such that $C_w L = \{0\}$ for all $w \in W \setminus c$ and $C_y L \neq \{0\}$ for certain $y \in c$. In particular, for the irreducible module generated by $\pi_t(C_{w_0}\theta_\rho C'_{w_0})$, the associated two-sided cell is c_0.

Another main result concerns one-dimensional representations of H_t:

- if $t(\theta_\alpha) = q$ for each simple root $\alpha \in P$ and if $\sum_{w \in W_0} q^{l(w)} \neq 0$ in the ground field k, then both $\pi_t(C_{w_0} H C'_{w_0})$ and $\pi_t(C'_{w_0} H C_{w_0})$ are two-sided ideals of H_t of dimension one.

The proofs of these results are enlightening calculations based on some properties of c_0 established in the author's previous works and the Macdonald formula. The results are interesting, since the hypothesis $\pi_t(C_{w_0}\theta_\rho C'_{w_0}) \neq 0$ can in principle be checked in the finite-dimensional algebra H_t.

References

[1] S. Arkhipov and R. Bezrukavnikov, *Perverse sheaves on affine flags and Langlands dual groups*, Israel J. Math. **170** (2009): 135–183.

[2] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979): 165–184.