数学研究及评论
Mathematical Research with Reviews
Issue 2 (2019) Art.13

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Proof of Gessel's γ-positivity conjecture
Electron. J. Combin. 23 (2016), no. 3, Paper 3.15, 9 pp.

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收稿日期：2019年10月23日

Let $\mathfrak{S}_n$ denote the set of all permutations of $[n] := \{1, 2, \ldots, n\}$. For a permutation $\pi \in \mathfrak{S}_n$, an index $i \in [n-1]$ is a *descent* of π if $\pi_i > \pi_{i+1}$. Denote by $\operatorname{des}(\pi)$ the number of descents of π.

Let $\mathcal{A}_n$ be the set of all polynomials in $f(s,t) = \sum_{k,l \geqslant 0}^{n} a_{k,l} s^k t^l \in \mathbb{Z}[s,t]$ with symmetries

$$a_{k,l} = a_{l,k} \quad \text{and} \quad a_{k,l} = a_{n-k,n-l} \quad \text{for all } k, l \geqslant 0.$$

The set $\mathcal{B}_n := \{(st)^i (1+st)^j (s+t)^{n-j-2i} | i, j \geqslant 0, j + 2i \leqslant n\}$ is a basis of $\mathcal{A}_n$.

Gessel conjectured that the joint distribution of descents and inverse descents on permutations, the *double Eulerian polynomial* $A_n(s,t)$, has nonnegative integral coefficients under the set $\mathcal{B}_n$:

Conjecture 1. Let

$$A_n(s,t) := \sum_{\pi \in \mathfrak{S}_n} s^{\operatorname{des}(\pi^{-1})+1} t^{\operatorname{des}(\pi)+1}.$$

Then, for $n \geq 1$,

$$A_n(s,t) = \sum_{\substack{i \geq 1, j \geq 0 \\ j+2i \leq n+1}} \gamma_{n,i,j} (st)^i (1+st)^j (s+t)^{n+1-j-2i},$$

where $\gamma_{n,i,j}$ are nonnegative integers.

One of the main results is to prove the conjecture. Using the recurrence of $A_n(s,t)$, the author first provided a new direct approach to get the recurrence relation of the coefficients $\gamma_{n,i,j}$ due to Visontai [1]. It does not follow immediately from the recurrence relation that $\gamma_{n,i,j}$ is a nonnegative integer. The author deduced the nonnegativity of $\gamma_{n,i,j}$ from the recurrence relation, and even more characterized completely when the coefficient $\gamma_{n,i,j}$ is positive.

Theorem 2. *For $n \geq 1$, the coefficients $\gamma_{n,i,j}$ are nonnegative. Moreover, the coefficient $\gamma_{n,i,j}$ is positive if and only if $i \geq 1, j \geq 0, 2i+j \leq n+1$ and $i(i+j) \geq n$.*

Using the same method, the author proved the similar result for the *generalized double Eulerian polynomial*

$$A_n^{(k)}(s,t) = \sum_{\pi \in \mathfrak{S}_n} s^{\operatorname{des}(\pi^{-1})+1} t^{\operatorname{des}(\pi\sigma)+1},$$

where $\sigma \in \mathfrak{S}_n$ is any fixed permutation with $\operatorname{des}(\sigma) = k-1$.

Theorem 3. *For $n \geq 1$, and $1 \leq k \leq n$, we have*

$$A_n^{(k)}(s,t) = \sum_{\substack{i \geq 1, j \geq 0 \\ j+2i \leq n+1}} \gamma_{n,i,j}^{(k)} (st)^i (1+st)^j (s+t)^{n+1-j-2i},$$

where $\gamma_{n,i,j}^{(k)}$ are nonnegative integers.

In the last part, the author considered the *Type B double Eulerian polynomial*

$$B_n(s,t) := \sum_{\sigma \in \mathfrak{B}_n} s^{\operatorname{des}_B(\pi^{-1})} t^{\operatorname{des}_B(\pi)},$$

where $\mathfrak{B}_n$ is the Type B Coxeter group whose elements are regarded as signed permutations on $[n]$, $\operatorname{des}_B(\pi) := \#\{i \in [n] : \pi_{i-1} > \pi_i\}$ and $\pi_0 = 0$.

Theorem 4. *For $n \geq 1$,*

$$B_n(s,t) = \sum_{\substack{i,j \geq 0 \\ j+2i \leq n}} \tilde{\gamma}_{n,i,j} (st)^i (1+st)^j (s+t)^{n-j-2i},$$

where $\widetilde{\gamma}_{n,i,}$ are nonnegative integers. Moreover, $\widetilde{\gamma}_{n,i,j}$ is positive if and only if $i,j \geq 0$, $2i+j \leq n$ and $2i(i+j+1)+j \geq n$.

To summarize, in this paper a unified method is used to prove the classic Gessel's conjecture about the joint distribution of descents and inverse descents on permutations, the generalization of the Gessel's conjecture and the Type B analog. The author directly proved the conjecture by means of the recurrence relation of coefficients. The question of finding combinatorial proofs of these positivity problems is still open.

References

[1] M. Visontai, Some remarks on the joint distribution of descents and inverse descents, *Electron. J. Combin.*, **20** (2013), #P52.