数学研究及评论
Mathematical Research with Reviews
Issue 2 (2019) Art.12

Xi, Nanhua (席南华)
Module structure on invariant Jacobians
Math. Res. Lett. 19 (2012), no. 3, 731–739.

评论员：陈晓煜 (上海师范大学，上海)
收稿日期：2019年10月20日

This paper is motivated by a conjecture of Yau, which arose from studying the classification of isolated hypersurface singularities in complex geometry ([2]). In [2], Yau and Mather pointed out that the classification of such singularities is completely determined by (the isomorphism class of) its moduli algebra

$$A(f) = \mathbb{C}[[x_1, \cdots, x_n]] / \left(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \right).$$

When studying the solvability of $L(f)$, which is the Lie algebra of derivations of $A(f)$, Yau made a conjecture on the set of highest weights of invariant Jacobians (see next paragraph for details).

Let K be a field, G be a group or a Lie algebra over K, and let V be a finite dimensional representation of G. Then $K[V]$, the set of polynomial functions of V is natural a G-module, and so is $K(V)$, the ring of "rational functions" on V. For a homogeneous function f in $K[V]$ or $K(V)$, the Jacobian $J(f)$ is the linear subspace of $K[V]$ or $K(V)$ spanned by the partial derivatives $\partial f/\partial x_i$, where $x_1, \ldots, x_d$ denote a basis for the linear functionals on V. More generally, the r-th Jacobian $J_r(f)$ is the linear subspace of $K[V]$ or $K(V)$ spanned by all r-th partial derivatives $\partial^r f/\partial x_1 \ldots \partial x_r$. Now let $G = SL_2(\mathbb{C})$, Yau's conjecture says that if $J(f)$ is G-invariant, then the highest weights of $J(f)$ is a subset of all highest weights in V^*.

In the paper under review, the author proves that if Kf is a G-module (i.e, f is G–semi-invariant) and $J(f)$ is G-invariant, then $J(f)$ is a quotient G-module of $Kf \otimes V$. The key step is constructing a map

$$\begin{aligned} \varphi : Kf \otimes V &\to J(f), \\ f \otimes e_i &\mapsto \partial f/\partial x_i, \end{aligned}$$

which turns out to be a G-module homomorphism. In particular, if f is G-invariant, then $J(f)$ is a quotient G-module of V. Now suppose that G is a connected, semisimple algebraic group over an algebraically closed field K of characteristic zero. Let f be homogeneous of degree at least 3, and suppose that $J(f)$ is G-invariant. G. Kempf has shown that there is an invariant polynomial g in $K[V]$ such that $\deg g = \deg f$ and $J(g) = J(f)$; see [3]. Using this result, the present paper shows that if $J_r(f)$ is G-invariant, then $J_r(f)$ is a quotient G-module of $S^r(V)$. In particular, this solves and generalizes Yau's conjecture.

One can form a Hopf algebraic version of the map φ above, with G replaced by quasi-triangular Hopf algebra, and $k[V]$ replaced by the "braided version" of $S^r(V)$; see [1] for details. The result should be useful for investigating representations of quasi-triangular Hopf algebras. Certainly, a natural problem is:

Can we find a noncommutative geometric version of Yau's conjecture?

This still remains open.

References

[1] X. Y. Chen, Quasi-triangular Hopf algebras and invariant Jacobians. Science China Math. 60 (2017) 421–430.

[2] J. N. Mather, S. S.-T. Yau, Classification of isolated hypersurface singularities by their moduli algebras. Invent Math. 69 (1982) 243–251.

[3] G. R. Kempf, Jacobians and invariants. Invent. Math. 112 (1993) 315-321.