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An upper bound for the second Neumann eigenvalue on Riemannian manifolds
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Let $\Omega \subset \mathbb{R}^n\,(n \geq 2)$ be a bounded domain with $\partial\Omega \in C^2$, let $\lambda \in \mathbb{R}$ and consider the Dirichlet eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

It is well known that there exists a non-decreasing sequence of eigenvalues for the above problem:

$$0 \leq \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \nearrow \infty$$

Let us first state the Faber-Krahn inequality which is one of the oldest isoperimetric inequalities for all kinds of eigenvalues.

Theorem 1. (**Faber-Krahn's inequality**) (see [5, 6]) For a bounded domain $\Omega \subset \mathbb{R}^n$, the first Dirichlet eigenvalue satisfies

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$$

with equality if and only if $\Omega = \Omega^*$. Here Ω^* is a round ball in $\mathbb{R}^n$ with the same volume as Ω.

The Faber-Krahn inequality is also valid for more general manifolds. We start by briefly introducing some general setting for the sake of convenience.

Let (M, g) be an n-dimensional complete Riemannian manifold, and Ω a bounded domain in M. Denote by $|\Omega|_M$ and $|\partial\Omega|_M$ the n-dimensional volume of Ω and the $(n-1)$-dimensional Hausdorff measure of $\partial\Omega$, and by $\mathrm{diam}(\Omega)$ the diameter of Ω. For a fixed $\kappa \in \mathbb{R}$, let $\mathbb{M}_\kappa$ be an n-dimensional simply-connected space form of constant sectional curvature κ, and Ω_q^* be the geodesic ball centered at q in $\mathbb{M}_\kappa$ with the same volume as Ω. Define the generalized sine function sn_κ on $\mathbb{R}$ by

$$\mathrm{sn}_\kappa(r) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} r, & \text{if } \kappa > 0, \\ r, & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} r, & \text{if } \kappa < 0. \end{cases}$$

Theorem 2. ([7]) Let Ω be any bounded domain in M, and $\mathfrak{B}$ the associated geodesic ball in $\mathbb{M}_\kappa$ satisfying

$$|\Omega| = |\mathfrak{B}|. \tag{1}$$

Suppose further that Ω satisfies $|\Omega| < |\mathbb{M}_\kappa|$ in the case of $\kappa > 0$. If, for all such Ω in M, equality (1) implies the isoperimetric inequality

$$|\partial\Omega| \geq |\partial\mathfrak{B}| \tag{2}$$

with equality in (2) if and only if Ω is isometric to $\mathfrak{B}$, then we also have, for every bounded domain Ω in M, that equality (1) implies the inequality for the first Dirichlet eigenvalue

$$\lambda_1(\Omega) \geq \lambda_1(\mathfrak{B})$$

and equality holds if and only if Ω is isometric to $\mathfrak{B}$.

Now consider the Neumann eigenvalue problem of the Laplacian Δ on a bounded domain $\Omega \subset M$ (M is the Euclidean space $\mathbb{R}^n$, or more generally, is some curved spaces):

$$\begin{cases} -\Delta u = \mu u & \text{in} \quad \Omega, \\ \frac{\partial}{\partial\nu} u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

where ν denotes the outer normal to $\partial\Omega$, and $\frac{\partial}{\partial\nu}u$ denotes the outer normal derivative of u along $\partial\Omega$. It is well-known that there exists a non-decreasing and discrete sequence of Neumann eigenvalues of (3), denoted by

$$\{\mu_i \mid 0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \leq \cdots \nearrow \infty\}.$$

Here $\mu_0 = 0$ since $u_0 = 1$ is precisely a Neumann eigenfunction of Δ corresponding to 0 for the domain Ω. The first non-zero Neumann eigenvalue μ_1 can be characterized by the Rayleigh quotient

$$\mu_1(\Omega) = \inf\left\{\frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega |u|^2\,dx}, \quad u \in C^\infty(\Omega)\backslash\{0\} \text{ with } \int_\Omega u\,dx = 0\right\}$$

Next we introduce the Szegö-Weinberger inequality which is a counterpart to the first non-zero Neumann eigenvalue of the Faber-Krahn inequality.

Theorem 3. (**Szegö-Weinberger's inequality**, see [4]) Let Ω be a bounded domain in $\mathbb{R}^n$. Then the first non-zero Neumann eigenvalue of Ω satisfies

$$\mu_1(\Omega) \le \mu_1(\Omega^*) \tag{4}$$

with equality holding if and only if $\Omega = \Omega^*$. Here Ω^* is defined as before.

This result was first proved by Szegö [3] in case $n = 2$, and was later generalized to high dimensions by Weinberger [4]. The Szegö-Weinberger inequality (4) also holds for bounded domains in hyperbolic space $\mathbb{H}^n$, and in a hemisphere of $\mathbb{S}^n$, see for example [1].

It is natural to ask if one can extend the Szegö-Weinberger inequality to the case of more general manifolds with certain curvature conditions.

In the paper under review, the author studies the upper bounds for the first non-zero Neumann eigenvalue of problem (3) on a bounded domain in Riemannian manifolds, and gets a comparison result for the first non-zero Neumann eigenvalue of the Laplacian on domains in Riemannian manifolds with that of a geodesic ball of the same volume in some appropriate space form. This actually provides a new way of estimating eigenvalues on bounded domains in the curved spaces. More precisely, the author obtains the following comparison theorem.

Theorem 4. Assume the sectional curvature of M is bounded from above by κ, and the Ricci curvature is bounded from below by $(n-1)K$. If $\kappa > 0$, assume further that there exists some strongly convex closed set, written as $\mathrm{hull}(\Omega)$, containing Ω and satisfying the following two conditions:

$$D := \mathrm{diam}(\Omega) = \mathrm{diam}(\mathrm{hull}(\Omega)) < \min\left\{\frac{\pi}{2\sqrt{\kappa}}, \text{ injectivity radius of } M\right\}$$

and

$$|\,\mathrm{hull}(\Omega)|_M \le \frac{|\mathbb{M}_\kappa|_{\mathbb{M}_\kappa}}{2},$$

then

$$\mu_1(\Omega) \leq \left(\frac{\mathrm{sn}_K(D)}{\mathrm{sn}_\kappa(D)}\right)^{2n-2} \mu_1\left(\Omega_q^*\right). \tag{5}$$

Remark 1. In the space form $\mathbb{M}_\kappa$, the ratio $\mathrm{sn}_K(D)/\mathrm{sn}_\kappa(D)$ equals to 1 due to $K = \kappa$, and then the inequality (5) reduces to the classical Szegö-Weinberger inequality on space forms, see [1, 3, 4]. So it can be viewed as a generalized Sezgö-Weinberger type inequality. Besides, the author showed that the hypotheses of Theorem 4 are necessary.

As a direct application of Theorem 4, the author also obtains the following result.

Corollary 1. Let M be an n-dimensional Hadamard manifold with Ricci curvature bounded from below by $(n-1)K$. Then

$$|\Omega|_M^{\frac{2}{n}} \mu_1(\Omega) \leq \left(\frac{\mathrm{sn}_K(D)}{D}\right)^{2n-2} \omega_n^{\frac{2}{n}} \mu_1\left(B_1\right),$$

where B_1 and ω_n denote an unit ball in $\mathbb{R}^n$ and the volume of B_1, respectively.

The proof of Theorem 4 combines the transplanting and averaging method, similar to that of the PPW inequality for the curved spaces by Edelen [2]. Precisely, one can transplant the eigenfunctions of Ω_q^* to Ω via spherical symmetrizations, and then use the averaging of the Rayleigh quotients for test functions to estimate $\mu_1(\Omega)$ from above.

Remark 2. One has some similar comparison theorems for Steklov eigenvalues. The reader is referred to [8] for more details about these results.

There are reasons to believe that the conclusion of Theorem 4 should also include a characterization of the equality case in (5). Finally, we raise the following question:

Is it true that the equality holds in (5) if and only if $\Omega = B$? Here B is some geodesic ball in M with the same volume as Ω.

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