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*Smooth solutions to the $L_p$ dual Minkowski problem*
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The classical Minkowski problem asks for necessary and sufficient conditions on a given measure so that it is the surface area measure of a convex body. The influence of this problem reaches far beyond convex geometric analysis to fields such as PDE, differential geometry, and functional analysis. In particular, special cases of the Minkowski problem include the problem of prescribing Gaussian curvature and a Monge-Ampère equation.

Since the classical Minkowski problem, many Minkowski type problems have been introduced and extensively studied, notably, the $L_p$ Minkowski problem. The $L_p$ Minkowski problem is the problem of prescribing $L_p$ surface area measure which was introduced by Lutwak [5]. When the given measure has a density function, the $L_p$ Minkowski problem reduces to an arguably much harder Monge-Ampère equation. In fact, two critical cases of the $L_p$ Minkowski problem, which are the logarithmic Minkowski problem ($p = 0$) and the centro-affine Minkowski problem ($p = -n$), still remain open. The $L_p$ Minkowski problem when $p > 1$ was solved by Lutwak [5] when the given measure is even and by Chou and Wang [2] without the evenness assumption. Other important contributions to critical cases of the $L_p$ Minkowski problem can be found in the paper under review.

In a groundbreaking work [3], Huang, Lutwak, Yang, and Zhang proposed a new family of geometric measures called dual curvature measures, which are dual to Federer's curvature measures in the classical Brunn-Minkowski theory, and posed the dual Minkowski problem—the problem of prescribing dual curvature measures. The dual Minkowski problem includes problems such as the Aleksandrov problem and the logarithmic Minkowski problem as special cases. The problem quickly becomes the center of attention leading to a lot of terrific works.

A recent surprising result by Lutwak, Yang, and Zhang [6] revealed a unified family of geometric measures, called the $(p, q)$th dual curvature measures, that contains all the aforementioned measures as special cases. This new family of geometric measures suggests that different theories such as the classical Brunn-Minkowski theory, the $L_p$ Brunn-Minkowski theory and the dual Brunn-Minkowski theory which naturally arise in the journey of exploring measures, invariants, inequalities involving convex bodies could all turn out to be a part of the new $L_p$ dual Brunn-Minkowski theory. The following $L_p$ dual Minkowski problem was posed in [6].

**Problem 1.1** *(The $L_p$ dual Minkowski problem) Given a nonzero finite Borel measure $\mu$ on the unit sphere $S^{n-1}$ and real numbers $p, q$, what are the necessary and sufficient conditions such that $\mu$ is equal to the $(p, q)$th dual curvature measure of some convex body $K$ containing the origin in its interior?*

The existence and uniqueness of smooth solution to the Monge-Ampère type equation of Problem 1.1 were obtained by Huang and Zhao [4] by using a method of continuity for prescribed smooth function $f$ and $p > q$. Since the $(p, q)$th dual curvature measures contain all the aforementioned geometric measures, the $L_p$ dual Minkowski problem, contains all the aforementioned Minkowski problems. When $p = 1$ and $q = n$, Problem 1.1 becomes the classical Minkowski problem. When $p = 0$ and $q = 0$, Problem 1.1 becomes the Aleksandrov problem, i.e., the problem of prescribing Aleksandrov's integral curvature. For $q = n$ and arbitrary $p$, Problem 1.1 reduces to the $L_p$ Minkowski problem. In particular, it contains the logarithmic Minkowski problem ($p = 0, q = n$), and the centro-affine Minkowski problem ($p = -n, q = n$). Both these problems are still unsolved. For $p = 0$ and an arbitrary $q$, Problem 1.1 reduces to the dual Minkowski problem.

Besides the fact that the existence part of many cases of $L_p$ dual Minkowski problem remain unsolved, results on the uniqueness part are even more scarce. Of all the aforementioned Minkowski problems, only the classical Minkowski problem and the Aleksandrov problem have their uniqueness problems completely settled. In fact, for the logarithmic Minkowski problem, uniqueness result was only found when the given measure is even in the planar case [1]. For the dual Minkowski problem, uniqueness result was only found when $q < 0$; see [7].

Using anisotropic Gauss-Kronecker curvature flows, the authors of the paper under review established the existence of smooth solutions of the $L_p$ dual Minkowski problem when $pq \geq 0$ and the given data is even. If $f \equiv 1$, the authors showed under some restrictions on $p$ and $q$ that the only even, smooth, uniformly convex solution is the unit ball.

The above results are very interesting and inspiring because they contain many interesting special cases. When $p = q = 0$, this is the even Aleksandrov problem. When $q = n$, this is the even $L_p$ Minkowski problem for $p \geq 0$. In particular, when $p = 0$, this is the even logarithmic Minkowski problem.

The main idea the authors used is to find a suitable functional and a suitable anisotropic Gauss-Kronecker curvature flow. The difficulties are to obtain uniform positive lower and upper bounds of the support function and principal curvatures along the flows, which is precisely the reason that a new flow has to be adopted.

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