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Li, Ai-Jun (李爱军); Huang, Qingzhong (黄卿忠); Xi, Dongmeng (席东盟)
New sine ellipsoids and related volume inequalities
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评论员：蔺友江 (重庆工商大学，重庆)
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Associated with each star body K in an n-dimensional Euclidean space $\mathbb{R}^n$, there is a unique ellipsoid $\Gamma_2 K$ which has the same moment of inertia as K does with respect to every 1-dimensional subspace of $\mathbb{R}^n$. This ellipsoid is called the *Legendre ellipsoid*, whose support function is defined by, for $x \in \mathbb{R}^n$,

$$h^2_{\Gamma_2 K}(x) = \frac{n+2}{V(K)} \int\limits_K |x \cdot y|^2 dy, \tag{1}$$

where $V(K)$ denotes the n-dimensional volume of K and $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$. Observing that Legendre ellipsoid is an object in the dual Brunn-Minkowski theory, Lutwak, Yang, and Zhang [1] introduced a new dual analog of the Legendre ellipsoid using the notion of L_2 surface area measure in the framework of the L_p Brunn-Minkowski theory. This ellipsoid is now called the *LYZ ellipsoid* $\Gamma_{-2} K$ whose radial function at $x \in \mathbb{R}^n \backslash \{o\}$ is given by

$$\rho^{-2}_{\Gamma_{-2} K} = \frac{1}{V(K)} \int\limits_{S^{n-1}} |x \cdot v|^2 dS_2(K, v), \tag{2}$$

where $S_2(K, \cdot)$ is the L_2 surface area measure of $K \in \mathcal{K}^n_o$ on the unit sphere S^{n-1}.

In the paper under review, the authors define the *sine ellipsoid* of the Legendre ellipsoid and the LYZ ellipsoid as follows. For each star body K, they define the sine ellipsoid $\Lambda_2 K$ whose support function at $x \in \mathbb{R}^n$ is given by

$$h^2_{\Lambda_2 K}(x) = \frac{n+2}{V(K)} \int\limits_K [x,y]^2 dy, \tag{3}$$

where $[x, y]$ denotes the 2-dimensional volume of the parallelepiped spanned by x, y. For each convex body K, they define the sine ellipsoid $\Lambda_{-2} K$ whose radial function at $x \in \mathbb{R}^n \backslash \{o\}$ is given by

$$\rho^{-2}_{\Lambda_{-2} K}(x) = \frac{1}{V(K)} \int\limits_{S^{n-1}} [x,v]^2 dS_2(K,v). \tag{4}$$

These four ellipsoids, specified by support functions (1), (3), or radial functions (2), (4), are closely related to the Pythagorean relation and duality. More precisely, $\Lambda_{-2}K$ and $\Lambda_2 K$ have Pythagorean relations with $\Gamma_{-2}K$ and $\Gamma_2 K$, respectively; $\Lambda_{-2}K$ and $\Gamma_{-2}K$ have duality relations with $\Lambda_2 K$ and $\Gamma_2 K$, respectively. Moreover, several volume inequalities and the valuation properties are obtained for these two new ellipsoids.

The results in this paper are very interesting. Firstly, these two new sine ellipsoids and their volume inequalities enrich the L_p Brunn-Minkowski theory and the dual Brunn-Minkowski theory. Secondly, the operation $|x \cdot y|$ of $x, y \in \mathbb{R}^n$ in (1) and (2) is related to the cosine transform, and $[x, y]$ in (3) and (4) is related to the sine transform. Both transforms can been generalized and unified to the L_p cosine transform on Grassmann manifolds (see [2]). Thirdly, the open problems provided in the last part of this paper are very important and are worthy of thinking deeply and studying further.

References

[1] E. Lutwak, D. Yang, G. Zhang *A new ellipsoid associated with convex bodies*, Duke Math. J. 104 (2000), 375-390.

[2] A.-J. Li, D. Xi, G. Zhang *Volume inequalities of convex bodies from cosine transforms on Grassmann manifolds*, Adv. Math. 304 (2017), 494-538.