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Continuity and minimization of spectrum related with the periodic Camassa-Holm equation

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In the paper under review, the authors considered the eigenvalue problem

$$y'' = \frac{1}{4}y + \lambda m(t)y, \tag{1}$$

with the periodic boundary condition

$$y(0) = y(1), \quad y'(0) = y'(1). \tag{2}$$

This problem is related to finding spatially periodic solutions for the Camassa-Holm equation. Let $\mathcal{C}$ be the space of continuous functions on $[0,1]$ and $\mathcal{C}^- = \{m \in \mathcal{C} : m(t) \leq 0, m(t) \not\equiv 0\}$. The authors proved that the k-th eigenvalue $\lambda_k(m)$ of (1)-(2) is continuous in $m \in \mathcal{C}^-$, where $k \geq 0$, $\mathcal{C}^-$ is considered as a subset of the Lebesgue space $\mathcal{L}^1 = \mathcal{L}^1([0,1])$ and $\mathcal{C}^-$ inherits the weak topology of $\mathcal{L}^1$. The proof is based on monotonicity and continuity results of $\hat{\lambda}_k(q)$ [2] and a relationship between $\lambda_k(m)$ and $\hat{\lambda}_k(q)$, where $\hat{\lambda}_0(q) > \hat{\lambda}_1(q) \geq \hat{\lambda}_2(q) > \cdots > \hat{\lambda}_{2k-1}(q) \geq \hat{\lambda}_{2k}(q) > \cdots$ are all the eigenvalues of $y'' = \lambda y + q(t)y$ with (2).

The authors then proved the continuity of the principal eigenvalue in measures. Let

$$\mu \in \mathcal{M}_0^- = \{\nu \in \mathcal{M}_0 : \nu(t) \text{ is decreasing on } [0,1], \text{ and } \nu \neq 0\},$$

where $\mathcal{M}_0 = (\mathcal{C}, \|\cdot\|_\infty)^*$ and $\nu \in \mathcal{M}_0$ is normalized as $\nu(0+) = 0$. The authors considered the measure differential equation

$$dy^\bullet = \frac{1}{4} y dt + \lambda y d\mu(t), \tag{3}$$

with the periodic boundary condition

$$y(0) = y(1), \quad y^\bullet(0) = y^\bullet(1). \tag{4}$$

Using a minimization characterization of the principal eigenvalue $\tilde{\lambda}_0(\mu)$ of (3)–(4) and the continuous dependence of solutions of initial value problems of measure differential equations in measures [1], the authors proved that $\tilde{\lambda}_0(\mu)$ is continuous in $\mu \in \mathcal{M}_0^-$, where $\mathcal{M}_0^-$ inherits the weak* topology on $\mathcal{M}_0$.

Finally, the authors gave the explicit solution of a minimization problem

$$\mathbf{L}(r) = \inf\{\lambda_0(m) : m \in \mathcal{C}^- \cap S_1[r]\} \equiv \inf\{\lambda_0(m) : m \in \mathcal{C}^- \cap B_1[r]\}, \tag{5}$$

where $r \in (0, \infty)$, $S_1[r] = \{q \in \mathcal{L}^1 : \|q\|_1 = r\}$ and $B_1[r] = \{q \in \mathcal{L}^1 : \|q\|_1 \leq r\}$. Note that $B_1[r]$ lacks compactness even in weak topology. As a result, (5) can not be attained by any $m \in \mathcal{C}^- \cap B_1[r]$. The authors' idea is first to extend the minimization problem as follows

$$\tilde{\mathbf{L}}(r) = \inf\{\tilde{\lambda}_0(\mu) : \mu \in \mathcal{M}_0^- \cap S_0[r]\} \equiv \inf\{\tilde{\lambda}_0(\mu) : \mu \in \mathcal{M}_0^- \cap B_0[r]\},$$

where $S_0[r] = \{\mu \in \mathcal{M}_0 : \|\mu\|_{\mathbf{V}} = r\}, B_0[r] = \{\mu \in \mathcal{M}_0 : \|\mu\|_{\mathbf{V}} \leq r\}$, and $\|\mu\|_{\mathbf{V}} = \sup\{\sum_{i=0}^{n-1} |\mu(t_{i+1}) - \mu(t_i)| : 0 = t_0 < t_1 < \cdots < t_n = 1, n \in \mathbb{N}\}$. Note that $\tilde{\mathbf{L}}(r)$ can be attained by some measure $\mu_r \in \mathcal{M}_0^- \cap S_0[r]$. Then they showed that for $r > 0$, $\tilde{\mathbf{L}}(r) = \frac{\tanh \frac{1}{4}}{r}$ and the lower bound is attained by $\mu = -r\delta_a, a \in [0, 1]$. For the ordinary differential equation (1), the authors obtained the same solution of the minimization problem (5): $\mathbf{L}(r) = \frac{\tanh \frac{1}{4}}{r}$.

In particular, the lower bound of the principal eigenvalue is optimal and more applicable, as is solved in the setting of MDEs. The results obtained in this paper are interesting to the scholars on spectral theory of ODEs (MDEs), as well as those focusing on the shallow water equations.

References

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