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Some improved inequalities related to Vizing's conjecture
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The concept of domination in graph theory has its motivations from real world problems such as resource relocation. A vertex v in a graph is said to be *dominated* by another vertex u if there is an edge from u to v in the graph. The *domination number* $\gamma(G)$ of a graph G is the minimum number of vertices required to dominate all other vertices of G. It is a well-known NP-hard problem to precisely determine the domination number of a graph. Extensive research has been devoted to establishing lower bounds for the domination number.

In 1968, V. G. Vizing conjectured that $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$, where $G\Box H$ is the *Cartesian product* of two graphs G and H. The Cartesian product is a useful method to construct larger graphs from smaller ones (such as the *hypercube graphs*). Many mathematicians have worked on Vizing's conjecture and obtained some partial results since the conjecture was proposed. Although it has been verified for special families of graphs or graphs satisfying certain conditions, the conjecture remains open today for arbitrary graphs.

On the other hand, many results similar to Vizing's conjecture have been discovered. In particular, Wu [5] showed that $\gamma_R(G\Box H) \geq \gamma(G)\gamma(H)$, where γ_R is a variant of the domination number γ defined as follows.

Given a graph G with vertex set V, let $f : V \to \{0, 1, 2\}$ be a function, which is equivalent to a partition of V into three mutually disjoint subsets V_0, V_1, and V_2, where $V_i = \{v \in V : f(v) = i\}$. We say that f is a *Roman dominating function* if every vertex in V_0 is dominated by some vertex in V_2. The *Roman domination number* $\gamma_R(G)$ is the minimum value of $\sum_{v \in V} f(v)$ among all Roman dominating functions f. This concept was implicitly given by ReVelle–Rosing [2] and Stewart [3], and formally introduced by Cockayne et al. [1], with the following alternative description.

Imagine that zero, one or two Roman legions were stationed at each given location. A location is considered to be *secure* if at least one legion is stationed there. An unsecured location can be secured by sending a legion to it from an adjacent location. Emperor Constantine the Great (c. 272 AD–337 AD) decreed that this cannot be done if it would leave any location unsecured, or in other words, at least two legions must be stationed at a location before one of them can be sent away. What is the minimum number of legions required to make sure all locations can be secured in the above way? This is exactly the Roman domination number of the underlying graph.

In the interesting paper under review, Li-Dan Pei, Xiang-Feng Pan, and Fu-Tao Hu established the lower bound $\gamma(G \Box H) \geq \frac{1}{4}\gamma_R(G)\gamma_R(H)$ for the domination number of the Cartesian product of any two graphs G and H, using the Roman domination numbers of G and H. It is known that $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$ for any graph G, and when the second inequality is an equality, G is called a *Roman graph*. Thus this new result implies Vizing's conjecture for Roman graphs. It also improves a previous result by Yero [6], which asserts that $\gamma(G \Box H) \geq \frac{1}{3}\gamma(G)\gamma_R(H)$, in certain cases.

The second important result of this paper is the following lower bound for the Roman domination number of the Cartesian product of two graphs G and H, assuming G and H are not both empty:

$$\gamma_R(G \Box H) \geq \gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}.$$

This improves the result $\gamma_R(G \Box H) \geq \gamma(G)\gamma(H)$ by Wu [5]. It is similar to the lower bound $\gamma(G \Box H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}$ obtained by Suen and Tarr [4] as an improvement of the so-called Clark–Suen inequality $\gamma(G \Box H) \geq \frac{1}{2}\gamma(G)\gamma(H)$, and is better in some cases.

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