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*The monotonicity and log-behaviour of some functions related to the Euler gamma function*
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The monotonicity and log-behavior are classic topics of functions and sequences. In this paper, the author proves the monotonicity and/or log-behaviors of some combinatorial sequences, the most important ones being involved with Euler Gamma functions and Riemann Zeta functions.

The most essential references of this paper are [1] and [2], from which the author develops similar analytic techniques and extends their results to more classes of combinatorial sequences.

Following are some key definitions in this paper.

**Monotonicity** As is well known, a sequence is monotone if and only if it is increasing or decreasing, i.e., $z_{n+1} \geq z_n$ for all $n \geq 1$ or $z_{n+1} \leq z_n$ for all $n \geq 1$.

**Log-Behavior** Let $\{z_n\}_{n\geq 0}$ be a sequence of positive numbers. It is called *log-concave* (respectively, *log-convex*) if $z_{n-1}z_{n+1} \leq z_n^2$ (respectively, $z_{n-1}z_{n+1} \geq z_n^2$ for all $n \geq 1$. It is equivalent to the fact that the sequence $\{z_{n+1}/z_n\}_{n\ geq 0}$ is decreasing (respectively, increasing).

**Infinitely Log-Monotonic** Define an operator $R$ on a sequence $\{z_n\}_{n\geq 0}$ by

$$R\{z_n\}_{n\geq 0} = \{x_n\}_{n\geq 0}$$

where $x_n = z_{n+1}/z_n$. The sequence $\{z_n\}_{n\geq 0}$ is called *infinitely log-monotonic* is the sequence $R^r\{z_n\}_{n\geq 0}$ is log-concave for all positive odd $r$ and is log-convex for all non-negative even $r$.

**Completely Monotonic** A function $f(x)$ is said to be *completely monotonic* on an interval $I$ if $f(x)$ has derivatives of all orders on $I$ that alternate successively in sign, i.e.

$$(-1)^n f^{(n)}(x) \geq 0$$

for all $x \in I$ and for all $n \geq 0$.

**Logarithmically Completely Monotonic** A positive function $f(x)$ is said to be *logarithmically completely monotonic* on an interval $I$ if $\log f(x)$ satisfies

$$(-1)^n [\log f(x)]^n \geq 0$$

for all $x \in I$ and for all $n \geq 0$.

One of the main results in this paper is Theorem 1.1.

**Theorem 1.1.** Let $N$ be a positive number. If $f(x)$ is a positive increasing log-convex function for $x \geq N$ and $f(N) \leq 1$, then $\sqrt[x]{f(x)}$ is strictly increasing on $(N, \infty)$.

The proof is quite elementary and only college calculus is needed. This result is "on the one hand" in the abstract where the author states that a criterion for the monotonicity of the function $\sqrt[x]{f(x)}$ is given, which is a continuous analogue of a result of Wang and Zhu.

Then the paper focus on the log-behavior of the sequence $\sqrt[n]{z_n}$. It starts from a known formula involving Gamma function and finally proves the conjecture in [2].

**Conjecture 1.2.** The function $\theta(x) = \sqrt[x]{2\zeta(x)\Gamma(x+1)}$ is log-concave on $(6, \infty)$.

With similar techniques, the log-concave properties of quite a lot sequences are proved, e.g.,

$$\{\sqrt[n]{(-1)^{n-1}B_{2n}}\}_{n\geq 1}, \quad \{\sqrt[n]{\frac{1}{2n+1}\binom{2n}{n}}\}_{n\geq 1}, \quad \{\sqrt[n]{\binom{2n}{n}}\}_{n\geq 1}, \quad \{\sqrt[n]{\binom{3n}{n}}\}_{n\geq 1},$$

$$\{\sqrt[n]{\binom{4n}{n}}\}_{n\geq 1}, \quad \{\sqrt[n]{\binom{5n}{n}}\}_{n\geq 1}, \quad \{\sqrt[n]{\binom{5n}{2n}}\}_{n\geq 1}, \quad \{\sqrt[n]{C_p(n)}\}_{n\geq 2}.$$

Then with a result of the connection between logarithmically completely monotonic functions and infinite log-monotonicity in [1], the author applies this connection to prove infinite log-monotonicity of several combinatorial sequences.

This paper is an extension and generalization of [1] and [2] where only elementary analytic methods and properties of Gamma functions are needed to derive and prove a set of properties for certain combinatorial classes. The main results are the proof of Theorem 1.1 and Conjecture 1.2, the application of similar analytic techniques to prove log-behavior for a broader class of sequences, and the application of results in [1] to prove the infinitely log-monotonic property of several sequences. In general, this paper is interesting, clear and straightforward, using elementary methods to find a wider class of sequences satisfying certain monotonic or log-monotonic properties. For sure even more sequences with these properties can be found and proved with existing techniques.

## References


[1] W.Y.C. Chen, J.J.F. Guo and L.X.W. Wang, *Infinitely log-monotonic combinatorial sequences*, Adv. Appl. Math. **52** (2014), 99-120.

[2] W.Y.C. Chen, J.J.F. Guo and L.X.W. Wang, *Zeta functions and the log-behavior of combinatorial sequences*, Proc. Edinb. Math. Soc. **58**(3) (2015), 637-651.