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New congruences involving products of two binomial coefficients

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令p为大于3的素数，Wolstenholme[10]于1862年建立了著名的同余式:

$$\frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p} \equiv 1 \ (mod\ p^3).$$

潘颢与孙智伟[3]于2006年证明了对任给的$d = 0, \ldots, p$, 有

$$\sum_{k=0}^{p-1}\binom{2k}{k+d} \equiv \left(\frac{p-d}{3}\right) (mod\ p),$$

其中$(-)$是Jacobi符号。此后孙智伟与Tauraso[9]进一步证明了

$$\sum_{k=0}^{p-1}\binom{2k}{k} \equiv \left(\frac{p}{3}\right) \ (mod\ p^2).$$

Mortenson[1, 2]于2003年借助Gross-Koblitz公式以及$p-$adic Γ函数证明了下面的同余式

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) \ (mod\ p^2), \ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \left(\frac{p}{3}\right) \ (mod\ p^2),$$

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k} \equiv \left(\frac{-2}{p}\right) \ (mod\ p^2), \ \sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k} \equiv \left(\frac{-1}{p}\right) \ (mod\ p^2).$$

在Gauss超几何级数的$p-$adic类式以及Calabi-Yau流形的启发下人们曾猜测出上述的超同余式。孙智伟[7]于2011年进一步证明了

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)-p^2E_{p-3}\ (mod\ p^3)$$

和

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)+p^2E_{p-3}\ (mod\ p^3),$$

其中$E_0,E_1,E_2,\ldots$由下式所确定：

$$\frac{2}{e^x+e^{-x}}=\sum_{n=0}^{\infty}E_n\frac{x^n}{n!}\ \left(|x|<\frac{\pi}{2}\right).$$

孙智伟在文[8]中又讨论了模p^3的$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^2}{(2k+1)16^k}$和$\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^2}{(2k+1)16^k}$的同余式，在文[7]中猜测并被孙智宏在文[5]中证明了下面三个同余式成立。

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(2k+1)27^k}\equiv\left(\frac{p}{3}\right)\ (mod\ p^2),$$

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(2k+1)64^k}\equiv\left(\frac{-1}{p}\right)-3p^2E_{p-3}\ (mod\ p^3),$$

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{(2k+1)432^k}\equiv\left(\frac{p}{3}\right)\ (mod\ p^2).$$

此外还有和$\sum_{k=1}^{\lfloor 3p/4\rfloor}a_k$相关的同余式，比如孙智伟在文[6]中证明了对任意的奇素数p有

$$\sum_{k=1}^{\lfloor\frac{3}{4}p\rfloor}\frac{(-1)^{k-1}}{k}\equiv\sum_{k=1}^{(p-1)/2}\frac{1}{k2^k}\ (mod\ p).$$

潘颢和孙智伟[4]还证明了对任意模4余1的素数有下面的超同余式成立。

$$\sum_{k=0}^{\lfloor\frac{3}{4}p\rfloor}\frac{\binom{2k}{k}}{4^k}\equiv\left(\frac{2}{p}\right)\ (mod\ p^2).$$

本文作者在上述结论的基础上证明了如下结果。

定理1. 对任意奇素数p，有

(i)

$$\sum_{k=0}^{\lfloor 3p/4\rfloor}\frac{\binom{2k}{k}^2}{16^k}\equiv\begin{cases}1(mod\ p^3) & \text{if } p\equiv 1\ (mod\ 4),\\ -1+p^2/\left(2\binom{(p-3)/2}{(p-3)/4}^2\right)(mod\ p^3) & \text{if } p\equiv 3\ (mod\ 4).\end{cases}$$

(ii) 对每一个$a = 2, 3, 4, ...,$ 有

$$\sum_{k=0}^{\lfloor \frac{3}{4}p^a \rfloor} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p^a}\right) \ (mod\ p^3).$$

定理2. 令p为大于3的素数，则有下面的同余式成立：

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \left(\frac{p}{3}\right) \frac{2^p+1}{3} \ (mod\ p^2),$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{6k}{3k}\binom{3k}{k}}{(2k+1)432^k} \equiv \left(\frac{p}{3}\right) \frac{3^p+1}{4} \ (mod\ p^2),$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{4k}{2k}\binom{2k}{k}}{(2k+1)64^k} \equiv \left(\frac{-1}{p}\right) 2^{p-1} \ (mod\ p^2).$$

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