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DYNAMICAL ELLIPTIC BETHE ALGEBRA, KZB EIGENFUNCTIONS, AND THETA-POLYNOMIALS

DANIEL THOMPSON° AND ALEXANDER VARCHENKO⋆

Abstract. Let $(\otimes_{j=1}^n V_j)[0]$ be the zero weight subspace of a tensor product of finite-dimensional irreducible $\mathfrak{sl}_2$-modules. The dynamical elliptic Bethe algebra is a commutative algebra of differential operators acting on $(\otimes_{j=1}^n V_j)[0]$-valued functions on the Cartan subalgebra of $\mathfrak{sl}_2$. The algebra is generated by values of the coefficient $S_2(x)$ of a certain differential operator $\mathcal{D} = \partial_x^2 + S_2(x)$, defined by V. Rubtsov, A. Silantyev, D. Talalaev in 2009. We express $S_2(x)$ in terms of the KZB operators introduced by G. Felder and C. Wieszerkowski in 1994. We study the eigenfunctions of the dynamical elliptic Bethe algebra by the Bethe ansatz method. Under certain assumptions we show that such Bethe eigenfunctions are in one-to-one correspondence with ordered pairs of theta-polynomials of certain degree. The correspondence between Bethe eigenfunctions and two-dimensional spaces, generated by the two theta-polynomials, is an analog of the non-dynamical non-elliptic correspondence between the eigenvectors of the $\mathfrak{gl}_2$ Gaudin model and the two-dimensional subspaces of the vector space $\mathbb{C}[x]$, due to E. Mukhin, V. Tarasov, A. Varchenko.

We obtain a counting result for, equivalently, certain solutions of the Bethe ansatz equation, certain fibers of the elliptic Wronski map, or ratios of theta polynomials, whose derivative is of a certain form. We give an asymptotic expansion for Bethe eigenfunctions in a certain limit, and deduce from that that the Weyl involution acting on Bethe eigenfunctions coincides with the action of an analytic involution given by the transposition of theta-polynomials in the associated ordered pair.

1. Introduction

A quantum integrable system is a vector space and a family of commuting linear operators called Hamiltonians. The problem is to find eigenvectors and eigenvalues.

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The eigenvectors may be interpreted as the states of the quantum integrable system and the eigenvalues as the observables.

If the vector space is a space of functions, then the Hamiltonians are differential or difference operators.

Representation theory is a source of interesting quantum integrable systems. One considers a quantum group with a commutative subalgebra and a module over the quantum group. Then the action of the commutative subalgebra on the module provides a quantum integrable system on the corresponding vector space. The eigenvectors of the operators of the commutative subalgebra are constructed with the help of the other operators of the quantum group. This method to construct the eigenvectors is called the Bethe ansatz method.

Typical examples of such quantum integrable systems are the Gaudin, XXX, XXZ, XYZ chains.

In this paper we consider the quantum integrable system in which the vector space is a space of functions and the Hamiltonians are differential operators. That quantum integrable system was constructed by Rubtsov, Silantyev, Talalaev in 2009 and called the *dynamical elliptic Gaudin model*.

Let $V = \otimes_{j=1}^{n} V_j$ be a tensor product of $\mathfrak{sl}_N$-modules, $V[0]$ the zero weight subspace, $z_1, \dots, z_n, \tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$. Let $\mathrm{Fun}_{\mathfrak{sl}_N}(V[0])$ be the space of $V[0]$-valued functions on the Cartan subalgebra of $\mathfrak{sl}_N$. Given these data we follow V. Rubtsov, A. Silantyev, D. Talalaev in [RST] and introduce a commutative algebra of linear differential operators acting on $\mathrm{Fun}_{\mathfrak{sl}_N}(V[0])$, which we call the *dynamical elliptic Bethe algebra.*[1] The algebra is given in the form of a differential operator

$$\mathcal{D} = \partial_x^N + \sum_{j=1}^{N} S_j(x)\partial_x^{N-j} \tag{1.1}$$

with respect to the variable x, where each coefficient $S_j(x)$ is a differential operator acting on $\mathrm{Fun}_{\mathfrak{sl}_N}(V[0])$ and depending on x as a parameter. We call the operator $\mathcal{D}$ the *universal differential operator* or the *RST-operator*. The dynamical elliptic Bethe algebra $\mathcal{B}^V(z_1, \dots, z_n, \tau)$ is the algebra generated by the identity operator and the operators $\{S_j(x) \mid x \in \mathbb{C},\ j = 1, \dots, N\}$.

Our Theorem 4.1 says that the Bethe algebra is doubly periodic,

$$\mathcal{D}(x + k + l\tau) = \mathcal{D}(x), \qquad k, l \in \mathbb{Z},$$

Theorem 4.4 says that the Bethe algebra is conjugated by some explicit operator, if some of the complex numbers $z_1, \dots, z_n$ are shifted by elements of the lattice $\mathbb{Z}+\tau\mathbb{Z}$.

Let $\Psi \in \mathrm{Fun}_{\mathfrak{sl}_N}(V[0])$ be an eigenfunction of the Bethe algebra. Then

$$S_j(x)\Psi = B_j(x)\Psi, \qquad j = 1, \dots, N,$$

where $B_j(x)$ is a scalar function of x of eigenvalues of the operator $S_j(x)$. This construction assigns to Ψ a scalar differential operator with respect to the variable

[1] In fact in [RST] the authors consider the tensor products of $\mathfrak{gl}_N$-modules, while in this paper we restrict ourselves to tensor products of $\mathfrak{sl}_N$-modules

x,

$$\mathcal{D}_\Psi = \partial_x^N + \sum_{j=1}^N B_j(x)\partial_x^{N-j}, \tag{1.2}$$

called the *fundamental differential operator* of the eigenfunction Ψ. We have $\mathcal{D}_\Psi(x+k+l\tau) = \mathcal{D}_\Psi(x)$ for $k,l\in\mathbb{Z}$.

The correspondence $\Phi\to\mathcal{D}_\Psi$ between eigenfunctions of the commutative dynamical elliptic Bethe algebra and ordinary differential operators of special form is in the spirit of the geometric Langlands correspondence. An example of a non-dynamical non-elliptic correspondence of that type see in [MTV1, MTV2], where an eigenvector Ψ of the non-dynamical $\mathfrak{gl}_N$ Gaudin model corresponds to a differential operator, whose kernel consists of polynomials only, and that polynomial kernel defines a point in the Grassmannian of N-planes in the vector space $\mathbb{C}[x]$.

In this paper we study the relation $\Phi\to\mathcal{D}_\Psi$ for the Bethe eigenfunctions in the case of the tensor product of irreducible $\mathfrak{sl}_2$-modules. In that case

$$\mathcal{D} = \partial_x^2 + S_2(x) \qquad \text{and} \qquad \mathcal{D}_\Psi = \partial_x^2 + B_2(x).$$

In Lemma 4.11 we show that the universal differential operator $\partial_x^2 + S_2(x)$ is invariant with respect to the natural action of the $\mathfrak{sl}_2$ Weyl group $W=\{\mathrm{id}, s\}$. Hence the Weyl group acts on the set of eigenfunctions of the dynamical elliptic Bethe algebra, and the eigenfunctions Ψ and $s(\Psi)$ have the same fundamental differential operators.

In Theorem 4.9 we show that

$$S_2(x,z,\tau) = -2\pi i H_0(z,\tau) - \sum_{s=1}^n \Big[H_s(z,\tau)\rho(x-z_s,\tau) + c_2^{(s)}\rho'(x-z_s,\tau)\Big], \tag{1.3}$$

where $H_j(z,\tau)$, $j=0,\dots,n$, are the KZB operators, introduced by G. Felder and C. Wieszerkowski in [FW] to describe the differential equations for conformal blocks on the elliptic curve $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$. The operator c_2 in (1.3) is the quadratic central element in $U(\mathfrak{sl}_2)$, and $\rho=\theta'/\theta$ is the logarithmic derivative of the normalized first Jacobi theta function, see (2.1).

Formula (1.3) shows that the dynamical elliptic Bethe algebra is generated by the KZB operators for the Lie algebra $\mathfrak{sl}_2$.

The construction of eigenfunctions of the KZB operators by the Bethe ansatz method was suggested in [FV1]. Let λ_{12} be the coordinate on the Cartan subalgebra of $\mathfrak{sl}_2$. Let $V=\otimes_{j=1}^n V_j$ be a tensor product of irreducible $\mathfrak{sl}_2$-modules with highest weights m_j, $j=1,\dots,n$. The zero weight subspace $V[0]$ is nontrivial if the sum $m_1+\dots+m_n$ is even, that is, the sum equals $2m$ for some integer $m\in\mathbb{Z}_{>0}$.

We introduces a certain *elliptic master function* $\Phi(\mu,t,z,\tau)$, depending on complex variables μ, $t=(t_1,\dots,t_m)$, $z=(z_1,\dots,z_n)$, τ, and introduce a certain meromorphic $V[0]$-valued function $\Psi(\lambda_{12},\mu,t,z,\tau)$ of λ_{12}, depending on the parameters t,z,μ,τ, see (5.3) and (10.1).

Given μ, z, τ, we consider the critical point equations for the master function with respect to the variables t,

$$\frac{\partial \Phi}{\partial t_j}(\mu, t, z, \tau) = 0, \qquad j = 1, \dots, m,$$

called the *Bethe ansatz equations.* By [FV1], if (μ, t, z, τ) is a solutions of the Bethe ansats equations, then the function $\Psi(\lambda_{12}, \mu, t, z, \tau)$ of λ_{12} is an eigenfunctions of the KZB operators,

$$H_0(z,\tau)\Psi(\lambda_{12}, \mu, t, z, \tau) = \frac{\partial \Phi}{\partial \tau}(\mu, t, z, \tau)\, \Psi(\lambda_{12}, \mu, t, z, \tau), \tag{1.4}$$
$$H_a(z,\tau)\, \Psi(\lambda_{12}, \mu, t, z, \tau) = \frac{\partial \Phi}{\partial z_a}(\mu, t, z, \tau)\, \Psi(\lambda_{12}, \mu, t, z, \tau), \qquad a = 1, \dots, n,$$

see Theorem 5.1. The eigenfunction $\Psi(\lambda_{12}, \mu, t, z, \tau)$ has a periodicity property:

$$\Psi(\lambda_{12} + 1, \mu, t, z, \tau) = e^{\pi i \mu} \Psi(\lambda_{12}, \mu, t, z, \tau).$$

Thus, given z, τ we have a family of meromorphic eigenfunctions of the elliptic dynamical Bethe algebra depending on the parameters (μ, t) such that (μ, t, z, τ) solve the Bethe ansatz equations.

The fundamental differential operator of the eigenfunction $\Psi(\lambda_{12}, \mu, t, z, \tau)$ is denoted by $\mathcal{D}_{(\mu,t,z,\tau)}$. Formulas (1.3) and (1.4) allow us to give the following formula for $\mathcal{D}_{(\mu,t,z,\tau)}$. Let

$$y(x) = \prod_{i=1}^{m} \theta(x - t_i, \tau), \qquad u(x) = e^{\pi i \mu x} y(x) \prod_{s=1}^{n} \theta(x - z_s, \tau)^{-m_s/2}.$$

Then

$$\mathcal{D}_{(\mu,t,z,\tau)} = \big(\partial_x + (\ln u)'\big)\big(\partial_x - (\ln u)'\big), \tag{1.5}$$

where $'$ is the derivative with respect to x, see Theorem 5.3.

Having this formula, we restrict ourselves to the case $V = (\mathbb{C}^2)^{\otimes 2m}$, study the kernels of the fundamental differential operators $\mathcal{D}_{(\mu,t,z,\tau)}$, and the dependence of the kernels on (μ, t).

We introduce the space of theta-polynomials of degree m and show, under certain conditions, that the kernel of $\mathcal{D}_{(\mu,t,z,\tau)}$ is generated by two functions $u_1(x), u_2(x)$ of the form,

$$u_1(x) = f/\sqrt{\mathrm{Wr}(f,g)}, \qquad u_2(x) = g/\sqrt{\mathrm{Wr}(f,g)}, \tag{1.6}$$

where f, g are theta-polynomials of degree m, and $\mathrm{Wr}(f,g) = fg' - f'g$ is the Wronskian of the functions f and g, see Theorems 9.7 and 9.8.

Under certain conditions, we show that the Bethe eigenfunctions $\Psi(\lambda_{12}, \mu, t, z, \tau)$ are in one-to-one correspondence with equivalence classes of ordered pairs (f, g) of theta-polynomials of degree m, see Theorems 9.7, 10.2, and 10.4.

We define the *analytic involution* on the set of ordered pairs of theta-polynomials by the formula $(f, g) \mapsto (g, f)$. The analytic involution induces an involution on the set of Bethe eigenfunctions. We prove that the analytic involution on the set of Bethe eigenfunctions coincides with the action of the Weyl involution s, see Theorem

10.3, see also [MV2, Section 5], where a non-dynamical non-elliptic version of this result had been established.

Under certain conditions, we show that the differential operators $\mathcal{D}_{(\mu,t,z,\tau)}$ are in one-to-one correspondence with the unordered pairs $(\Psi, s(\Psi))$ of Bethe eigenfunctions, see Theorem 10.4. That means that two Bethe eigenfunctions $\Psi(\lambda_{12},\mu,t,z,\tau)$ and $\Psi(\lambda_{12},\mu',t',z,\tau)$ have the same eigenvalues for every element of the dynamical elliptic Bethe algebra $\mathcal{B}^V(z_1, \dots, z_{2m},\tau)$, if and only if either $\Psi(\lambda_{12},\mu,t,z,\tau) = \Psi(\lambda_{12},\mu',t',z,\tau)$ or $s(\Psi(\lambda_{12},\mu,t,z,\tau)) = \Psi(\lambda_{12},\mu', t',z,\tau)$ up to proportionality.

In the proofs we use asymptotics of the solutions of Bethe ansatz equations when $\mu \to \infty$, see Theorems 11.2, 12.2, Lemma 12.4.

As a byproduct of that asymptotic study we obtain some counting results. For example, in Theorem 12.2, we show that in the limit $\mu \to \infty$, the number of Bethe eigenfunctions with given μ, z, τ equals $\dim(\mathbb{C}^2)^{\otimes 2m}[0] = \binom{2m}{m}$, and the leading terms of asymptotics of those Bethe eigenfunctions form the standard weight basis of $(\mathbb{C}^2)^{\otimes 2m}[0]$.

We also count the number of ratios of theta-polynomials of degree m with prescribed zeros of the derivative. More precisely, fix a fundamental parallelogram $\Lambda \subset \mathbb{C}$ of the lattice $\mathbb{Z} + \tau\mathbb{Z}$ acting on $\mathbb{C}$. Consider the ratio F of two theta-polynomials of degree m with m simple poles in Λ. Then the function F can be written uniquely in the form

$$F = g/f, \qquad f = \prod\nolimits_{j=1}^{m} \theta(x - t_j, \tau), \tag{1.7}$$

where $t = (t_1, \dots, t_m)$ is a point of Λ^m/S_m with distinct coordinates and g is a theta-polynomial of degree m. The derivative $F' = \mathrm{Wr}(f,g)/f^2$ can be written uniquely in the form

$$F' = e^{-2\pi i \mu x} \frac{\prod_{a=1}^{2m} \theta(x - z_a, \tau)}{\prod_{j=1}^{m} \theta(x - t_j, \tau)^2} \tag{1.8}$$

for some $\mu \in \mathbb{C}$ and $z = (z_1, \dots, z_{2m}) \in \Lambda^{2m}/S_{2m}$. By our assumption the numerator and denominator of this ratio have no common zeros. Let Λ' be the interior of Λ. Assume that $z = (z_1, \dots, z_{2m})$ belongs $(\Lambda')^{2m}/S_{2m}$ and has distinct coordinates. Then there exists $N > 0$, such that for any $\mu \in \mathbb{C}$ with $|\mu| > N$ there exist exactly $\binom{2m}{m}$ functions $F(x)$ as in (1.7) with the derivative as in (1.8), up to proportionality, see Theorem 12.1.

The paper is organized as follows. In Section 2 we collect useful formulas on theta functions. In Section 3 the RST-operator and KZB operators are introduced. First properties of the RST-operator are discussed in Section 4. In particular, we prove the double periodicity of the RST-operator. We describe the transformations of the RST-operator under the shifts of the parameters $z_1, \dots, z_n$ by elements of the lattice $\mathbb{Z}+\tau\mathbb{Z}$. We describe the Laurent expansion of the RST-operator in Theorem 4.6 and express the coefficient $S_2(x)$ in terms of the KZB operators. The Bethe ansatz for $\mathfrak{sl}_2$ is discussed in Section 5. We restrict ourselves to the case $V = (\mathbb{C}^2)^{\otimes 2m}$ in Section 6. In Section 7 the theta-polynomials are introduced. In Section 8 we study the Wronskian equation $\mathrm{Wr}(f,g) = h$ for theta-polynomails f, g, h of degrees $m, m, 2m$, respectively. In particular, we present a theorem from an unpublished paper [BMV] by L. Borisov, E. Mukhin, A. Varchenko, which says that given f, h, then, under certain conditions, there exists a unique g satisfying the equation $\mathrm{Wr}(f, g) = h$, see Theorem 8.4. In Section 9 the interrelations between solutions of the Bethe ansatz equations and ordered pairs of theta-polynomials are studied. In Section 10 a formula for Bethe eigenfunctions is given and the properties of Bethe eigenfunctions are discussed. In Section 11 we introduce the elliptic Wronski map and describe the asymptotic behavior of its fibers in the limit $\mathrm{Im}\,\mu \to \infty$, see the important for us Theorem 11.2. We discuss the applications of Theorem 11.2 in Section 12, in particular, we give a proof of Theorem 10.3 that the analytic and Weyl involutions coincide on Bethe eigenfunctions.

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2. Preliminaries

2.1. **Theta functions.** Our notations on theta functions follow [FV2]. Let $z, q \in \mathbb{C}$ with $|q| < 1$. Denote

$$(z;q) = \prod_{j=0}^{\infty}(1 - zq^j).$$

Let $z = e^{2\pi i x}$ and $q = e^{2\pi i \tau}$. The first Jacobi theta function is defined by the formula

$$\theta_1(x,\tau) = ie^{\pi i(\tau/4 - x)}(z;q)(q/z;q)(q;q).$$

It is an entire holomorphic odd function such that

$$\theta_1(x + n + m\tau, \tau) = (-1)^{m+n}e^{-\pi i m^2\tau - 2\pi i m u}\theta_1(x,\tau), \qquad m, n \in \mathbb{Z}.$$

It obeys the heat equation

$$4\pi i \frac{\partial}{\partial \tau}\theta_1(x,\tau) = \theta_1''(x,\tau),$$

where $' = d/dx$. Introduce

$$\begin{aligned} \theta(x,\tau) &= \frac{e^{-\pi i \tau/4}}{2\pi (q;q)^3}\theta_1(x,\tau) = -\frac{e^{-\pi i x}}{2\pi i}(z;q)(q/z;q)(q;q)^{-2} \\ &= \frac{\sin(\pi x)}{\pi}(qz;q)(q/z;q)(q;q)^{-2}. \end{aligned} \tag{2.1}$$

Then

$$\begin{aligned} \theta(x + n + m\tau, \tau) &= (-1)^{m+n}e^{-\pi i m^2\tau - 2\pi i m u}\theta(x,\tau), \qquad m, n \in \mathbb{Z}, \\ \theta'(0,\tau) &= 1. \end{aligned}$$

Denote

$$\begin{aligned} \rho(x,\tau) &= \frac{\theta'(x,\tau)}{\theta(x,\tau)}, \qquad \sigma(x,w,\tau) = \frac{\theta(x+w,\tau)}{\theta(x,\tau)\theta(w,\tau)}, \\ \varphi(x,w,\tau) &= \partial_x \sigma(w,-x,\tau), \qquad \eta(x) = \rho(x)^2 + \rho'(x). \end{aligned}$$

In the sequel, we will often omit the τ argument.

2.2. **Collected formulas.** We have for any $m,n\in\mathbb{Z}$, we have

$$\begin{aligned}
\rho(x+n+m\tau) &= \rho(x)-2\pi i m,\\
\rho'(x+n+m\tau) &= \rho'(x),\\
\sigma(x+n+m\tau,w) &= e^{-2\pi i m w}\sigma(x,w),\\
\varphi(x+n+m\tau,w) &= e^{2\pi i m w}\varphi(x,w),\\
\varphi(x,w+n+m\tau) &= e^{2\pi i m x}\big(\varphi(x,w)+2\pi i m\,\sigma(w,-x)\big),\\
\eta(x+n+m\tau,w) &= \eta(x,w)-4\pi i m\,\rho(x)+(2\pi i m)^2.
\end{aligned}$$

If $a_1,\dots,a_n,\ z_1,\dots,z_n\in\mathbb{C}$ and $\sum_{j=1}^n a_j=0$, then the function

$$f(x)=\sum_{j=1}^n a_j\rho(x-z_j,\tau)$$

is doubly periodic:

$$f(x+n+m\tau)=f(x),\qquad m,n\in\mathbb{Z}.$$

The function $\rho'(x,\tau)$ is even, while $\theta(x,\tau)$ and $\rho(x,\tau)$ are odd. We also have the identities

$$\begin{aligned}
\sigma(-x,-w) &= -\sigma(x,w)\\
\varphi(-x,-w) &= \varphi(x,w).
\end{aligned}$$

We have the following Laurent series expansions about $x=0$,

$$\begin{aligned}
\theta(x) &= x+\mathcal{O}(x^3),\\
\rho(x) &= x^{-1}+\mathcal{O}(x),\\
\sigma(x,w) &= x^{-1}+\rho(w)+\mathcal{O}(x),\\
\varphi(x,w) &= x^{-2}+\mathcal{O}(1),\\
\eta(x) &= \mathcal{O}(1).
\end{aligned}$$

Lemma 2.1. We have the identities

$$\begin{aligned}
\varphi(x,w) &= \sigma(w,-x)(\rho(x-w)-\rho(x)),\\
\varphi(x,0) &= -\rho'(x).
\end{aligned}$$

Proof. The first identity is obtained by taking the logarithmic derivative:

$$\frac{\partial_x\sigma(w,-x)}{\sigma(w,-x)}=-\frac{\partial_x\theta(w-x)}{\theta(w-x)}+\frac{\partial_x\theta(-x)}{\theta(-x)},$$

while the second can by seen taking Laurent expansion about $w=0$:

$$\sigma(w,-x)(\rho(x-w)-\rho(x))=(w^{-1}+\mathcal{O}(1))(-\rho'(x)w+\mathcal{O}(w^2)).$$

□

Lemma 2.2. Assume that $z_1,z_2\in\mathbb{C}$ do not differ by an element of $\mathbb{Z}+\tau\mathbb{Z}$. Then

$$\tfrac{\sigma(x-z_1,w)\sigma(x-z_2,-w)}{\sigma(z_1-z_2,-w)}+\rho(x-z_2)-\rho(x-z_1)=\rho(w)-\rho(w-(z_1-z_2)),\tag{2.2}$$

$$\sigma(x,w)\sigma(x,-w)=\rho'(w)-\rho'(x).\tag{2.3}$$

Proof. The difference,

$$\frac{\sigma(x-z_1,w)\sigma(x-z_2,-w)}{\sigma(z_1-z_2,-w)}+\rho(x-z_2)-\rho(x-z_1)-\rho(w)+\rho(w-(z_1-z_2)),$$

is a doubly periodic function, both in the variable x and in the variable w. The difference is entire in both variables as well, hence it must be a constant. Evaluating at $w=x-z_2$, this constant is seen to be zero. The second identity is proved similarly. □

Remark. Formula (2.2) is a reformulation of the following identity:

$$\frac{\prod_{1\leqslant j<l\leqslant 3}\theta(x_j+x_l)}{\theta(x_1+x_2+x_3)\prod_{j=1}^{3}\theta(x_j)}-\sum_{j=1}^{3}\rho(x_j)+\rho(x_1+x_2+x_3)=0,$$

which can be obtained from (2.2) by a change of variables.

Lemma 2.3. The function

$$f(x)=\rho(x-z_1)\rho(x-z_2)+\rho(z_1-z_2)\rho(x-z_2)-\rho(z_1-z_2)\rho(x-z_1)$$

satisfies the identities

$$f(z_1)=f(z_2)=\eta(z_1-z_2).$$

Proof. Observe the Laurent expansion about $x=z_1$:

$$\begin{aligned}&\rho(x-z_1)\left(\rho(x-z_2)-\rho(z_1-z_2)\right)\\&\quad=\left((x-z_1)^{-1}+\mathcal{O}(x-z_1)\right)\left(\rho'(z_1-z_2)(x-z_1)+\mathcal{O}((x-z_1)^2)\right),\end{aligned}$$

so

$$f(z_1)=\rho'(z_1-z_2)+\rho(z_1-z_2)^2=\eta(z_1-z_2).$$

A similar argument shows $f(z_2)=\eta(z_2-z_1)=\eta(z_1-z_2)$. □

3. The RST-operator

3.1. **The RST-operator for $\mathfrak{gl}_N$.** Consider the complex Lie algebra $\mathfrak{gl}_N$ with standard basis e_{jl}, $j,l=1,\dots,N$. Let $\mathfrak{h}\subset\mathfrak{gl}_N$ be the Cartan subalgebra generated by the elements e_{jj}, $j=1,\dots,N$. Let $\lambda\in\mathfrak{h}$ with $\lambda=\lambda_1e_{11}+\dots+\lambda_Ne_{NN}$.

Let $z=\{z_1,\dots,z_n\}\subset\mathbb{C}$ be numbers such that no pairwise difference belongs to $\mathbb{Z}+\tau\mathbb{Z}$.

Let $V_1,\dots,V_n$ be $\mathfrak{gl}_N$-modules and $V=\otimes_{k=1}^{n}V_k$. For an element $g\in U(\mathfrak{gl}_N)$, we write $g^{(k)}=1\otimes\dots\otimes g\otimes\dots\otimes 1$, with the element g in the k-th factor. Let $V=\oplus_{\boldsymbol{\lambda}\in\mathfrak{h}^*}V[\boldsymbol{\lambda}]$ be the the weight decomposition, where $V[\boldsymbol{\lambda}]=\{v\in V\mid e_{jj}v=\boldsymbol{\lambda}(e_{jj})v \text{ for } j=1,\dots,N\}$. In particular,

$$V[0]=\{v\in V\mid e_{jj}v=0,\ j=1\dots,N\}.$$

Denote $\lambda_{jl}=\lambda_j-\lambda_l$,

$$\begin{aligned}e_{jj}^{+}(x)&=\sum\nolimits_{k=1}^{n}\rho(x-z_k)e_{jj}^{(k)},\\e_{jl}^{+}(x)&=\sum\nolimits_{k=1}^{n}\sigma(x-z_k,\lambda_{jl})e_{jl}^{(k)},\quad\text{for } j\neq l.\end{aligned}$$

Introduce the $N \times N$-matrix $\mathcal{L}(x) = (\mathcal{L}_{jl}(x))$,

$$\begin{aligned} \mathcal{L}_{jj}(x) &= e^{+}_{jj}(x) + \sum\nolimits_{l\neq j} \rho(\lambda_{jl}) e_{ll}, \\ \mathcal{L}_{jl}(x) &= e^{+}_{lj}(x), \qquad \text{for } j \neq l. \end{aligned}$$

Here e_{ll} acts on V as $e^{(1)}_{ll} + \cdots + e^{(n)}_{ll}$.

The *universal dynamical differential operator* (or the *RST-operator*) is defined by the formula

$$\mathcal{D} = \operatorname{cdet}(\delta_{jl}\partial_x - \delta_{jl}\partial_{\lambda_j} + \mathcal{L}_{jl}), \tag{3.1}$$

where, for an $N \times N$-matrix $X = (X_{jl})$ with noncommuting entries the *column determinant* is defined by the formula

$$\operatorname{cdet} X = \sum\nolimits_{\sigma \in S_N} (-1)^{\sigma} X_{\sigma(1),1} \dots X_{\sigma(N),N}.$$

Let us write

$$\mathcal{D} = \partial_x^N + \sum\nolimits_{j=1}^{N} S_j(x) \partial_x^{N-j}. \tag{3.2}$$

Recall the tensor product V and the zero-weight subspace $V[0] \subset V$. We interpret every coefficient $S_j(x)$ as a function of x with values in the space of differential operators in variables $\lambda_1, \dots, \lambda_N$ with coefficients in $\operatorname{End}(V)$.

Theorem 3.1 ([RST]). Fix $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau > 0$. Fix $z = \{z_1, \dots, z_n\} \subset \mathbb{C}$ so that that no pairwise difference belongs to $\mathbb{Z} + \tau\mathbb{Z}$. Then for $x \in \mathbb{C}$, and any $j = 1, \dots, N$, the operator $S_j(x)$ well-defines a differential operator in $\lambda_1, \dots, \lambda_N$ with coefficients in $\operatorname{End}(V[0])$. Moreover, for any $j, l \in \{1, \dots, N\}$, $u, v \in \mathbb{C}$, these differential operators $S_j(u), S_l(v)$ commute,

$$[S_j(u), S_l(v)] = 0.$$

3.2. **The RST-operator for $\mathfrak{sl}_N$ and Bethe algebra.** In this paper, we are interested in the $\mathfrak{sl}_N$ version of the RST-operator.

The Lie algebra $\mathfrak{sl}_N$ is a Lie subalgebra of $\mathfrak{gl}_N$. We have $\mathfrak{gl}_N = \mathfrak{sl}_N \oplus \mathbb{C}(e_{11} + \cdots + e_{NN})$, where $e_{11} + \cdots + e_{NN}$ is a central element. Let $V_1, \dots, V_n$ be $\mathfrak{sl}_N$-modules, thought of as $\mathfrak{gl}_N$-modules, where the central element $e_{11} + \cdots + e_{NN}$ acts by zero. Let $V = \otimes_{k=1}^n V_k$ be the tensor product of the $\mathfrak{sl}_N$-modules. In this paper *we consider only such tensor products.*

We identify the algebra of functions on the Cartan subalgebra of $\mathfrak{sl}_N$ with the algebra of functions in the variables $\lambda_1, \dots, \lambda_N$, which depend only on the differences $\lambda_{jl} = \lambda_i - \lambda_l$. Indeed, the Cartan subalgebra of $\mathfrak{sl}_N$ consists of elements $\lambda = \lambda_1 e_{11} + \cdots + \lambda_N e_{NN}$ with $\lambda_1 + \cdots + \lambda_N = 0$. Such elements are determined uniquely by the differences λ_{jl} of the coordinates.

Denote by $\operatorname{Fun}_{\mathfrak{sl}_N}(V[0])$ the space of $V[0]$-valued meromorphic functions in the variables $\lambda_1, \dots, \lambda_N$, which depend only on the differences $\lambda_{jl} = \lambda_i - \lambda_l$.

Each coefficient $S_j(x)$ of the RST-operator, associated with $V[0]$, defines a differential operator acting on $\operatorname{Fun}_{\mathfrak{sl}_N}(V[0])$. From now on *we consider the coefficients* $S_1(x)$, *…*, $S_N(x)$ *as a family of commuting differential operators on* $\operatorname{Fun}_{\mathfrak{sl}_N}(V[0])$, depending on the parameter x.

The commutative algebra $\mathcal{B} = \mathcal{B}^V(z_1, \dots, z_n, \tau)$ of operators on $\mathrm{Fun}_{\mathfrak{sl}_N}(V[0])$ generated by the identity operator and the operators $\{S_j(x) \mid j = 1, \dots, N,\ x \in \mathbb{C}\}$ is called the *dynamical elliptic Bethe algebra* of V.

3.3. **The KZB operators.** Introduce the following elements of $\mathfrak{gl}_N \otimes \mathfrak{gl}_N$:

$$\Omega_0 = \sum\nolimits_{k=1}^N e_{kk} \otimes e_{kk}, \qquad \Omega_{jl} = e_{jl} \otimes e_{lj} \quad \text{for } j \neq l, \qquad \Omega = \Omega_0 + \sum\nolimits_{j \neq l} \Omega_{jl}.$$

The KZB operators $H_0(z_1, \dots, z_n, \tau), \dots, H_n(z_1, \dots, z_n, \tau)$ as operators on the $V[0]$-valued functions on the Cartan subalgebra of $\mathfrak{sl}_N$ were introduced in [FW]. As differential operators on $\mathrm{Fun}_{\mathfrak{sl}_N}(V[0])$ they have the following form:

$$H_0(z, \tau) = \frac{1}{4\pi i} \sum_{k=1}^N \partial^2_{\lambda_k} + \frac{1}{4\pi i} \sum_{s,p} \Big[\frac{1}{2} \eta(z_s - z_p, \tau) \Omega_0^{(s,p)} - \sum_{j \neq l} \varphi(\lambda_{jl}, z_s - z_p, \tau) \Omega_{jl}^{(s,p)} \Big],$$

$$H_s(z, \tau) = -\sum\nolimits_{k=1}^N e_{kk}^{(s)} \partial_{\lambda_k} + \sum\nolimits_{p\,:\,p \neq s} \Big[\rho(z_s - z_p, \tau) \Omega_0^{(s,p)} + \sum\nolimits_{j \neq l} \sigma(z_s - z_p, -\lambda_{jl}, \tau) \Omega_{jl}^{(s,p)} \Big],$$

see these formulas in Section 3.3 of [JV].

By [FW] the operators $H_0(z, \tau), H_1(z, \tau), \dots, H_n(z, \tau)$ commute and

$$\sum\nolimits_{s=1}^n H_s(z, \tau) = 0. \tag{3.3}$$

3.4. **Center of $\mathfrak{gl}_N$.** Let $Z(x)$ be the following polynomial in the variable x with coefficients in $U(\mathfrak{gl}_N)$:

$$Z(x) \ = \ \mathrm{rdet} \begin{pmatrix} x - e_{11} & -e_{21} & \dots & -e_{N1} \\ -e_{12} & x + 1 - e_{22} & \dots & -e_{N2} \\ \dots & \dots & \dots & \dots \\ -e_{1N} & -e_{2N} & \dots & x + N - 1 - e_{NN} \end{pmatrix}. \tag{3.4}$$

The next statement was proved in [HU], see also [MNO, Section 2.11].

Theorem 3.2. The coefficients of the polynomial $Z(x) - x^N$ are free generators of the center of $U(\mathfrak{gl}_N)$. □

For example, the coefficient of x^{N-2} equals

$$\sum\nolimits_{1 \leqslant j < l \leqslant N} ((e_{jj} - j + 1)(e_{ll} - l + 1) - e_{lj} e_{jl}) \tag{3.5}$$

$$= \sum\nolimits_{1 \leqslant j < l \leqslant N} (e_{jj} e_{ll} - e_{lj} e_{jl}) + \sum\nolimits_{j=1}^N (j - 1) e_{jj} - \frac{N(N-1)}{2} \sum\nolimits_{j=1}^N e_{jj} + \mathrm{const}$$

$$= \sum\nolimits_{1 \leqslant j < l \leqslant N} (e_{jj} e_{ll} - e_{lj} e_{jl} + e_{ll}) - \frac{N(N-1)}{2} \sum\nolimits_{j=1}^N e_{jj} + \mathrm{const}$$

$$= \sum\nolimits_{1 \leqslant j < l \leqslant N} (e_{jj} e_{ll} - e_{jl} e_{lj} + e_{jj}) - \frac{N(N-1)}{2} \sum\nolimits_{j=1}^N e_{jj} + \mathrm{const}\,.$$

4. First properties of the Bethe algebra

4.1. **Double periodicity of the RST-operator.**

Theorem 4.1. The operator $\mathcal{D}$ satisfies

$$\mathcal{D}(x+k+l\tau)=\mathcal{D}(x), \qquad k,l\in\mathbb{Z},$$

so each $S_j(x)$ is a doubly periodic function in the variable x.

Proof. Clearly $\mathcal{D}(x+1)=\mathcal{D}(x)$. Let us study $\mathcal{D}(x+\tau)-\mathcal{D}(x)$. The diagonal part the matrix of $\mathcal{D}(x+\tau)-\mathcal{D}(x)$ is the matrix $-2\pi i\,\mathrm{diag}(e_{11},\dots,e_{NN})$. The off-diagonal (jl)-th entry of $\mathcal{D}(x+\tau)$ equals the off-diagonal (jl)-th entry of $\mathcal{D}(x)$ multiplied by $e^{2\pi i\lambda_{jl}}$.

Decompose $\mathcal{D}(x+\tau)$ into the sum of monomials

$$M_L(x+\tau):=\mathcal{D}_{l_1,1}(x+\tau)\mathcal{D}_{l_2,2}(x+\tau)\dots\mathcal{D}_{l_N,N}(x+\tau).$$

Assume that such a monomial has a factor $\mathcal{D}_{rr}(x+\tau)=\partial_x-\partial_{\lambda_r}+\mathcal{L}_{rr}(x)-2\pi i e_{rr}$ for some r.

Lemma 4.2. The presence of the term $-2\pi i e_{rr}$ in $\mathcal{D}_{rr}(x+\tau)$ does not contribute to the difference $\mathcal{D}(x+\tau)-\mathcal{D}(u)$.

Proof. In this monomial the element e_{rr} stays in front of the product

$$\mathcal{D}_{l_{r+1},r+1}(x+\tau)\mathcal{D}_{l_{r+2},r+2}(x+\tau)\dots\mathcal{D}_{l_N,N}(x+\tau).$$

Since $r\notin\{r+1,\dots,N,l_{r+1},\dots,l_N\}$, the element e_{rr} commutes with $\mathcal{D}_{l_{r+1},r+1}(x+\tau)\times\mathcal{D}_{l_{r+2},r+2}(x+\tau)\dots\mathcal{D}_{l_N,N}(x+\tau)$. Hence e_{rr} can be written as the last factor of that monomial. After that e_{rr} will act on the zero weight subspace by zero and the corresponding product will act by zero. □

Lemma 4.3. The operator ∂_{λ_r} commutes with each of the factors $e^{2\pi i\lambda_{r+1,l_{r+1}}},\dots,e^{2\pi i\lambda_{N,l_N}}$ appearing in the product $\mathcal{D}_{l_{r+1},r+1}(x+\tau)\dots\mathcal{D}_{l_N,N}(x+\tau)$.

Proof. The lemma follows from the fact that $r\notin\{r+1,\dots,N,l_{r+1},\dots,l_N\}$. □

By the first of the previous lemma we see that in order to prove the theorem we need to check that the factors $e^{2\pi i\lambda_{jl}}=\mathcal{D}_{jl}(x+\tau)/\mathcal{D}_{jl}(x)$ do not change the determinant. By the second of the previous lemma we see that those factors behave like in a standard commutative determinant (they are not being differentiated). Thus we need to check that in an arbitrary monomial $M_L(x+\tau)$ the total product of the factors $e^{2\pi i\lambda_{jl}}$ equals zero. But that follows from the fact that $l_1,\dots,l_N$ is a permutation of $1,\dots,N$. □

4.2. **Dependence upon z.**

Recall that $z=\{z_1,\dots,z_n\}$ is an n-tuple of complex numbers such that no pairwise difference belongs to $\mathbb{Z}+\tau\mathbb{Z}$. The RST-operator depends on z,τ. It is clear from the formulas of Section 2.2 that $\mathcal{D}(x,z_1,\dots,z_n,\tau)=\mathcal{D}(x,z_1,\dots,z_s+m,\dots,z_n,\tau)$ for any $m\in\mathbb{Z}$.

Theorem 4.4. We have

$$\mathcal{D}(x,z_1,\dots,z_s+\tau,\dots,z_n,\tau)=e^{-2\pi i\sum_l\lambda_l e_{ll}^{(s)}}\mathcal{D}(x,z_1,\dots,z_n,\tau)e^{2\pi i\sum_l\lambda_l e_{ll}^{(s)}}.$$

Proof. Let us calculate the effect of conjugating by $A = e^{-2\pi i \sum_l \lambda_l e_{ll}^{(s)}}$:

$$\begin{aligned} Ae_{jj}^{(s)}A^{-1} &= e_{jj}^{(s)}, \\ Ae_{jl}^{(s)}A^{-1} &= e^{2\pi i\lambda_{jl}}e_{jl}^{(s)}, \qquad j \neq l, \\ A\partial_{\lambda_j}A^{-1} &= \partial_{\lambda_j} - 2\pi i e_{jj}^{(s)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} A\mathcal{L}_{jj}(x,z)A^{-1} &= \mathcal{L}_{jj}(x,z), \\ A\mathcal{L}_{jl}(x,z)A^{-1} &= e^{2\pi i\lambda_{lj}}\sigma(x - z_s, \lambda_{lj})e_{lj}^{(s)} + \sum\nolimits_{k\neq s}\sigma(x - z_k, \lambda_{lj})e_{lj}^{(k)}. \end{aligned}$$

Let $\mathcal{D}_{jl}(x,z) = \delta_{jl}\partial_x - \delta_{jl}\partial_{\lambda_j} + \mathcal{L}_{jl}$. Then

$$\begin{aligned} A\,\mathcal{D}_{jj}(x,z)A^{-1} &= \partial_x - \partial_{\lambda_j} + 2\pi i e_{jj}^{(s)} + \mathcal{L}_{jj} = \mathcal{D}_{jj}(x,z)A^{-1} + 2\pi i e_{jj}^{(s)}, \\ A\,\mathcal{D}_{jl}(x,z)A^{-1} &= e^{2\pi i\lambda_{lj}}\sigma(x - z_s, \lambda_{lj})e_{lj}^{(s)} + \sum\nolimits_{k\neq s}\sigma(x - z_k, \lambda_{lj})e_{lj}^{(k)}. \end{aligned}$$

On the other hand, the formulas of Section 2.2 yield

$$\begin{aligned} \mathcal{L}_{jj}(x, z_1, \dots, z_s + \tau, \dots, z_n) &= \mathcal{L}_{jj}(x, z_1, \dots, z_n) + 2\pi i e_{jj}^{(s)}, \\ \mathcal{L}_{jl}(x, z_1, \dots, z_s + \tau, \dots, z_n) &= e^{2\pi i\lambda_{lj}}\sigma(x - z_s, \lambda_{lj})e_{lj}^{(s)} + \sum\nolimits_{k\neq s}\sigma(x - z_k, \lambda_{lj})e_{lj}^{(k)}, \end{aligned}$$

thus we have shown that for any j, l, $D_{jl}(x, z_1, \dots, z_s + \tau, \dots, z_n) = A\,\mathcal{D}_{jl}(x,z)A^{-1}$. This yields the theorem. □

Corollary 4.5. The two commutative algebras $\mathcal{B}^V(z_1, \dots, z_n, \tau)$ and $\mathcal{B}^V(z_1, \dots, z_s + \tau, \dots, z_n, \tau)$ of operators on $\mathrm{Fun}_{\mathfrak{sl}_N}(V[0])$ are conjugated by the operator $e^{-2\pi i \sum_l \lambda_l e_{ll}^{(s)}}$.

4.3. **Laurent decomposition of the RST-operator.**

Theorem 4.6. Let

$$\mathcal{D}(x,z,\tau) = \sum\nolimits_{j=1}^{N} \frac{c_{i,j}(z,\tau)}{(x - z_i)^j} + \mathcal{O}(1), \qquad i = 1, \dots, n,$$

be the Laurent expansion of $\mathcal{D}(x,z,\tau)$ at $x = z_i$, where $c_{i,j}(z,\tau)$ are operators on $\mathrm{Fun}_{\mathfrak{sl}_N}(V[0])$. Then

$$\mathcal{D}(x,z,\tau) = \sum\nolimits_{i=1}^{n}\sum\nolimits_{j=1}^{N} \frac{(-1)^{j-1}}{(j-1)!}\,\rho^{(j-1)}(x - z_i, \tau)\,c_{i,j}(z,\tau) \;+\; c_0(z,\tau),$$

where $\rho^{(j-1)}(x,\tau)$ is the $(j-1)$-st derivative of $\rho(x,\tau)$ with respect to x, and $c_0(z,\tau)$ is an operator on $\mathrm{Fun}_{\mathfrak{sl}_N}(V[0])$ independent of x.

Proof. First, choose a fundamental parallelogram Λ for $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ acting on $\mathbb{C}$ such that each z_i, $i = 1, \dots, n$, differs by an element of $\mathbb{Z} + \tau\mathbb{Z}$ from an element $\tilde{z}_i$ in the interior of Λ. Then the set of poles of $\mathcal{D}(x,z,\tau)$ in Λ with respect to x is $\{\tilde{z}_1, \dots, \tilde{z}_n\}$. Since $\mathcal{D}(x,z,\tau)$ is doubly periodic, the Laurent expansion of $\mathcal{D}(x,z,\tau)$ at $x = z_i$ has the same coefficients as the expansion at $x = \tilde{z}_i$.

We have

$$\sum_{i=1}^{n} c_{i,1}(z,\tau) = \int_{\partial\Lambda} \mathcal{D}(x,z,\tau)dx = 0.$$

Now the difference $\mathcal{D}(x) - \sum_{i=1}^{n}\sum_{j=1}^{N}\frac{(-1)^{j-1}}{(j-1)!}\rho^{(j-1)}(x-\tilde{z}_i,\tau)c_{i,j}$ is an entire doubly periodic function of x, thus the difference is a constant with respect to x. □

Corollary 4.7. The dynamical elliptic Bethe algebra $\mathcal{B}^V(z,\tau)$ is generated by the $nN+1$ elements $c_{i,j}(z,\tau)$ and $c_0(z,\tau)$ of Theorem 4.6. □

4.4. **The coefficients $S_1(x)$ and $S_2(x)$.**

Lemma 4.8. We have

$$S_1(x,z,\tau) = \sum\nolimits_{j=1}^{N}\mathcal{L}_{jj}(x,z,\tau) - \sum\nolimits_{j=1}^{N}\partial_{\lambda_j} = 0$$

as an operator on $\mathrm{Fun}_{\mathfrak{sl}_N}(V[0])$. □

Theorem 4.9. We have the following identity of operators on $\mathrm{Fun}_{\mathfrak{sl}_N}(V[0])$:

$$S_2(x,z,\tau) = -2\pi i H_0(z,\tau) - \sum\nolimits_{s=1}^{n}\Big[H_s(z,\tau)\rho(x-z_s,\tau) + c_2^{(s)}\rho'(x-z_s,\tau)\Big],$$

where $c_2 = \sum_{j<l}(e_{jj}e_{ll} - e_{jl}e_{lj} + e_{jj})$ is a central element by equation (3.5).

Corollary 4.10. Assume that $N=2$ and each V_s, $s=1,\dots,n$, is an irreducible $\mathfrak{sl}_2$-module. Then the dynamical elliptic Bethe algebra $\mathcal{B}^V(z,\tau)$ is generated by the KZB operators $H_0(z,\tau)$, …,$H_n(z,\tau)$.

Proof. Since all V_s are irreducible, each $(c_2)^{(s)}$ acts by a scalar on $\mathrm{Fun}_{\mathfrak{sl}_2}(V[0])$. □

Proof of Theorem 4.9. Denote

$$\begin{aligned} F(x,\lambda_{jl}) \;&:=\; S_2(x) + 2\pi i H_0(z) + \sum\nolimits_{s=1}^{n}\Big[H_s(z)\rho(x-z_s) + c_2^{(s)}\rho'(x-z_s)\Big] \\ &\quad -\sum\nolimits_{s\neq p}\rho(z_s-z_p)\rho(x-z_s)(c_1\otimes c_1)^{(s,p)}. \end{aligned}$$

It is immediate that $F(x,\lambda_{jl})$ is meromorphic in x, having poles of at most second order at the points $z_s+\mathbb{Z}+\tau\mathbb{Z}$, $s=1,\dots,n$, and holomorphic elsewhere. Similarly, $F(x,\lambda_{jl})$ is meromorphic in each varaible λ_{jl}, having poles of at most second order at the points $\mathbb{Z}+\tau\mathbb{Z}$, and holomorphic elsewhere.

We have

$$\begin{aligned} S_2(x) &= \sum\nolimits_{1\leqslant j<l\leqslant N}\Big[(\mathcal{L}_{jj}-\partial_{\lambda_j})(\mathcal{L}_{ll}-\partial_{\lambda_l}) - \mathcal{L}_{lj}\mathcal{L}_{jl} + \mathcal{L}'_{ll}\Big] \\ &= \sum\nolimits_{1\leqslant j<l\leqslant N}\big[\partial_{\lambda_j}\partial_{\lambda_l} - \mathcal{L}_{jj}\partial_{\lambda_l} - \mathcal{L}_{ll}\partial_{\lambda_j} + \mathcal{L}_{jj}\mathcal{L}_{ll} - \mathcal{L}_{lj}\mathcal{L}_{jl} - \frac{\partial\mathcal{L}_{ll}}{\partial\lambda_j} + \frac{\partial\mathcal{L}_{ll}}{\partial x}\big]. \end{aligned}$$

We may expand this expression

$$\begin{aligned} S_2(x) &= -\frac{1}{2}\sum_j\partial_{\lambda_j}^2 + \sum_{j<l}\Big[\sum_s(-\rho(x-z_s)e_{jj}\partial_{\lambda_l} - \rho(x-z_s)e_{ll}\partial_{\lambda_j} + \rho'(x-z_s)e_{ll})^{(s)} \\ &\quad + \sum_{s,p}(\rho(x-z_s)\rho(x-z_p)e_{jj}\otimes e_{ll} - \sigma(x-z_s,\lambda_{jl})\sigma(x-z_p,-\lambda_{jl})e_{jl}\otimes e_{lj})^{(s,p)}\Big]. \end{aligned}$$

The function $S_2(x)$ is doubly periodic in the variable x by Theorem 4.1.

The coefficient of $(x-z_s)^{-2}$ in the Laurent expansion of $S_2(x)$ about z_s is $c_2^{(s)}$. Hence $F(x,\lambda_{jl})$ has no second order poles in the variable x.

The coefficient of $(x-z_s)^{-1}$ in $S_2(x)$ is

$$\sum\nolimits_{j<l}\ \Big[-(e_{jj}\partial_{\lambda_l}+e_{ll}\partial_{\lambda_j})^{(s)}+\sum\nolimits_{p:p\neq s}\big[\rho(z_s-z_p)(e_{jj}\otimes e_{ll}+e_{ll}\otimes e_{jj})$$
$$-\sigma(z_s-z_p,-\lambda_{jl})e_{jl}\otimes e_{lj}-\sigma(z_s-z_p,\lambda_{jl})e_{lj}\otimes e_{jl}\big]^{(s,p)}\Big],$$

which equals $-H_s(z)$. We have $\sum_s H_s(z)=0$ by (3.3). Hence the function

$$-\sum\nolimits_{s=1}^{n}H_s(z)\rho(x-z_s)$$

is doubly periodic with respect to x and has the same residues as $S_2(x)$. Now we may conclude that $F(x,\lambda_{jl})$ is an entire doubly periodic function in x. Thus it is a function only depending on the variables λ_{jl}.

Next, we may write

$$F(x,\lambda_{jl}) \;=\; \sum\nolimits_s c_2^{(s)}\rho'(x-z_s)-\sum\nolimits_{s\neq p}\rho(z_s-z_p)\rho(x-z_s)(c_1\otimes c_1)^{(s,p)}$$
$$+\sum\nolimits_s\big[\sum\nolimits_{j<l}-\rho(x-z_s)(e_{jj}\partial_{\lambda_l}+e_{ll}\partial_{\lambda_j})-\sum\nolimits_j\rho(x-z_s)e_{jj}\partial_{\lambda_j}\big]^{(s)}$$
$$+\sum\nolimits_{s\neq p}\sum\nolimits_{j<l}\Big[-\sigma(x-z_s,\lambda_{jl})\sigma(x-z_p,-\lambda_{jl})-\varphi(\lambda_{jl},z_s-z_p)$$
$$+\sigma(z_s-z_p,-\lambda_{jl})(\rho(x-z_s)-\rho(x-z_p))\Big](e_{jl}\otimes e_{lj})^{(s,p)}$$
$$+\sum_s\sum_{j<l}\Big[\Big(-\sigma(x-z_s,\lambda_{jl})\sigma(x-z_s,-\lambda_{jl})-\frac{1}{2}\varphi(\lambda_{jl},0)\Big)e_{jl}e_{lj}-\frac{1}{2}\varphi(\lambda_{jl},0)e_{lj}e_{jl}\Big]^{(s)}$$
$$+\sum\nolimits_{s\neq p}\sum\nolimits_{j<l}\rho(x-z_s)\rho(x-z_p)(e_{jj}\otimes e_{ll})^{(s,p)}$$
$$+\sum\nolimits_{s\neq p}\sum\nolimits_j\big[\frac{1}{4}\eta(z_s-z_p)+\rho(x-z_s)\rho(z_s-z_p)\big](e_{jj}\otimes e_{jj})^{(s,p)}$$
$$+\sum\nolimits_s\sum\nolimits_{j<l}\big[\rho(x-z_s)^2e_{jj}e_{ll}+\rho'(x-z_s)e_{ll}\big]^{(s)}+\sum\nolimits_s\sum\nolimits_j\frac{1}{4}\eta(0)(e_{jj}^2)^{(s)}.$$

The second line is

$$-\sum\nolimits_s\rho(x-z_s)\big[\sum\nolimits_{1\leqslant j<l\leqslant N}(e_{jj}\partial_{\lambda_l}+e_{ll}\partial_{\lambda_j})+\sum\nolimits_j e_{jj}\partial_{\lambda_j}\big]^{(s)}$$
$$=-\sum\nolimits_s\rho(x-z_s)c_1^{(s)}(\partial_{\lambda_1}+\cdots+\partial_{\lambda_N}),$$

and hence is zero.

By Lemma 2.2, the sum on the third and fourth lines is zero, while the fifth line is equal to

$$\sum_s\sum_{j<l}\big[\rho'(x-z_s)e_{jl}e_{lj}+\frac{1}{2}\varphi(\lambda_{jl},0)(e_{jj}-e_{ll})\big]^{(s)}$$
$$=\sum_s\sum_{j<l}\big[\rho'(x-z_s)e_{jl}e_{lj}\big]^{(s)}+\sum_{j<l}\frac{1}{2}\varphi(\lambda_{jl},0)(e_{jj}-e_{ll})=\sum_s\sum_{j<l}\big[\rho'(x-z_s)e_{jl}e_{lj}\big]^{(s)},$$

in $D_{\mathfrak{gl}}(V[0])$. This shows that, in fact, $F(x,\lambda_{jl})$ does not depend on λ_{jl} either.

Now, the expression for $F = F(x, \lambda_{jl})$ inside of $D_{\mathfrak{sl}}(V[0])$ reduces to

$$\sum_{s\neq p}\sum_{j<l}\Big[\rho(x-z_s)\rho(x-z_p)+\rho(z_p-z_s)\rho(z_p-x)+\rho(z_s-z_p)\rho(z_s-x)\Big](e_{jj}\otimes e_{ll})^{(s,p)}$$
$$+\sum_{s\neq p}\sum_{j}\frac{1}{4}\eta(z_s-z_p)(e_{jj}\otimes e_{jj})^{(s,p)}+\sum_{s}\sum_{j<l}\eta(x-z_s)(e_{jj}e_{ll})^{(s)}+\sum_{s}\sum_{j}\frac{1}{4}\eta(0)(e_{jj}^2)^{(s)}.$$

To complete the proof, we must show that $F = 0$. Let us view F as a $D_{\mathfrak{gl}}(V[0])$-valued function in the variable z_k, clearly regular at $z_k = x$ and $z_k = z_j$ for each $j \neq k$. We will show that F is a doubly periodic function in each variable z_k. It follows that F is entire in z_k, hence it does not depend on the choice of these points. Then we can degenerate to the case when all $z_k = 0$, which will show that the resulting operator acts on $V[0]$ by zero.

We write $F = F(z_k)$ to emphasize the dependence of F on the variable z_k. We have

$$F(z_k+\tau)-F(z_k)=\sum_{s\neq k}\Big[\sum_{j<l}\big[(2\pi i)^2-2\pi i\left(\rho(z_k-u)+\rho(z_k-z_s)\right)\big](e_{jj}\otimes e_{ll}+e_{ll}\otimes e_{jj})$$

$$+\frac{1}{2}\sum_{j}\big[(2\pi i)^2-4\pi i\rho(z_k-z_s)\big]e_{jj}\otimes e_{jj}\Big]^{(k,s)}+\sum_{j<l}\big[(2\pi i)^2-4\pi i\rho(z_k-x)\big](e_{jj}e_{ll})^{(k)}$$

$$=\sum_{s\neq k}\Big[\sum_{j\neq l}\big[(2\pi i)^2-2\pi i\left(\rho(z_k-x)\right)\big](e_{jj}\otimes e_{ll})+\frac{1}{2}\sum_{j}\big[(2\pi i)^2\big]e_{jj}\otimes e_{jj}\Big]^{(k,s)}$$

$$+\frac{1}{2}\sum_{j\neq l}\big[(2\pi i)^2-4\pi i\rho(z_k-x)\big](e_{jj}e_{ll})^{(k)}$$

$$-2\pi i\sum_{s\neq k}\rho(z_k-z_s)\Big[\sum_{j\neq l}(e_{jj}\otimes e_{ll}+\sum_{j}e_{jj}\otimes e_{jj}\Big]^{(k,s)}$$

$$=\Big[\sum_{j\neq l}\big[(2\pi i)^2-2\pi i\left(\rho(z_k-x)\right)\big](e_{jj}^{(k)}e_{ll}-(e_{jj}e_{ll})^{(k)})+\frac{1}{2}\sum_{j}(2\pi i)^2(e_{jj}^{(k)}e_{jj}-(e_{jj}^2)^{(k)})\Big]$$

$$+\frac{1}{2}\sum_{j\neq l}\big[(2\pi i)^2-4\pi i\rho(z_k-x)\big](e_{jj}e_{ll})^{(k)}-2\pi i\sum_{s\neq k}\rho(z_k-z_s)\big[c_1\otimes c_1\big]^{(k,s)}$$

$$=2\pi^2(c_1)^{2(k)}+\sum_{j\neq l}\big[(2\pi i)^2-2\pi i\left(\rho(z_k-x)\right)\big]e_{jj}^{(k)}e_{ll}$$
$$+\frac{1}{2}\sum_{j}(2\pi i)^2e_{jj}^{(k)}e_{jj}-2\pi i\sum_{s\neq k}\rho(z_k-z_s)\big[c_1\otimes c_1\big]^{(k,s)},$$

hence it is zero in $D_{\mathfrak{gl}}(V[0])$.

Now, evaluating at $x = z_1 = \cdots = z_n = 0$ using the Lemma 2.3, we obtain the following expression for F.

$$
\begin{aligned}
F &= \sum_{s\neq p}\sum_{j<l}\eta(0)(e_{jj}\otimes e_{ll})^{(s,p)} + \sum_{s\neq p}\sum_{j}\frac{1}{4}\eta(0)(e_{jj}\otimes e_{jj})^{(s,p)} \\
&\quad + \sum_{s}\sum_{j<l}\eta(0)(e_{jj}e_{ll})^{(s)} + \sum_{s}\sum_{j}\frac{1}{4}\eta(0)(e_{jj}^2)^{(s)} = \frac{1}{4}\eta(0)\big[(c_1)^2 + 2\sum_{j<l}e_{jj}e_{ll}\big],
\end{aligned}
$$

which is zero in $D_{\mathfrak{gl}}(V[0])$. □

4.5. **Weyl group invariance.** The Weyl group W acts on the Cartan subalgebra of $\mathfrak{sl}_N$ and on the space $V[0]$ in the standard way. Hence the Weyl group acts on $\mathrm{Fun}_{\mathfrak{sl}_N}(V[0])$ by the formula

$$s : \psi(\lambda) \mapsto s.(\psi(s^{-1}.\lambda)),$$

for $s \in W$, $\psi \in \mathrm{Fun}_{\mathfrak{sl}_N}(V[0])$. This extends to a Weyl group action on the space $\mathrm{End}(\mathrm{Fun}_{\mathfrak{sl}_N}(V[0]))$, where for $T \in \mathrm{End}(\mathrm{Fun}_{\mathfrak{sl}_N}(V[0]))$ and $s \in W$, the operator $s(T)$ is defined as the product sTs^{-1} of the three elements of $\mathrm{End}(\mathrm{Fun}_{\mathfrak{sl}_N}(V[0]))$.

Lemma 4.11. For the Lie algebra $\mathfrak{sl}_2$, the RST-operator $\mathcal{D}(x, z, \tau) \in \mathrm{End}(\mathrm{Fun}_{\mathfrak{sl}_2}(V[0]))$ is Weyl group invariant.

Proof. By [FW] the KZB operators $H_0(z, \tau), \dots, H_n(z, \tau)$ are Weyl group invariant. Now the lemma follows from Theorem 4.9. □

Corollary 4.12. All elements of the $\mathfrak{sl}_2$ dynamical elliptic Bethe algebra $\mathcal{B}^V(z, \tau)$ are Weyl group invariant.

The Weyl group invariance of the RST-operator for $\mathfrak{sl}_N$ will be discussed elsewhere.

5. The RST-operator and Bethe ansatz for $\mathfrak{sl}_2$

5.1. **Representations.** We consider the zero weight subspace $V[0]$ of the tensor product of $\mathfrak{sl}_2$ representations, where $V = \otimes_{s=1}^n V_{m_s}$ and V_{m_s} is the finite-dimensional irreducible representation with highest weight m_s, considered as an $\mathfrak{gl}_2$-module on which the central element $e_{11} + e_{22}$ acts by zero. The dimension of $V[0]$ is positive if the sum $\sum_{s=1}^n m_s$ is even. We denote this sum by $2m$,

$$\sum_{s=1}^{n} m_s = 2m. \tag{5.1}$$

5.2. **Bethe ansatz.** Let

$$\xi = \frac{\mu}{2}\alpha, \tag{5.2}$$

where $\mu \in \mathbb{C}$ and α is the simple root of $\mathfrak{sl}_2$, $\langle \alpha, e_{11}\rangle = 1$, $\langle \alpha, e_{22}\rangle = -1$.

The *elliptic master function* (see Section 5 of [FV1]) associated to $\mu \in \mathbb{C}$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ is the following function of μ, $t = (t_1, \dots, t_m)$, z, τ:

$$\Phi(\mu, t, z, \tau) = \frac{\pi i}{2}\mu^2\tau + 2\pi i\mu\left(\sum_{i=1}^m t_i - \sum_{s=1}^n \frac{m_s}{2} z_s\right) + \ 2\sum_{1\leqslant i<j\leqslant m} \ln\theta(t_i - t_j, \tau)$$
$$- \sum_{i=1}^m\sum_{s=1}^n m_s \ln\theta(t_i - z_s, \tau) + \sum_{1\leqslant s<r\leqslant n} \frac{m_s m_r}{2} \ln\theta(z_s - z_r, \tau). \tag{5.3}$$

The *Bethe ansatz equations* are the equations for the critical points of the master function $\Phi(\mu, t, z, \tau)$ with respect to the variables t,

$$2\pi i\mu + 2\sum_{j\neq i} \rho(t_i - t_j, \tau) - \sum_{s=1}^n m_s \rho(t_i - z_s, \tau) = 0, \qquad i = 1, \dots, m. \tag{5.4}$$

Theorem 5.1 ([FV1]). Let $(\mu^0, t_1^0, \dots, t_m^0, z_1^0, \dots, z_n^0, \tau^0)$ be a solution of the Bethe ansatz equations (5.4). Then there is a $V[0]$-valued meromorphic function of λ_{12}, denoted by $\Psi(\lambda_{12}, \mu^0, t^0, z^0, \tau^0)$, such that

$$H_a(z^0, \tau^0)\, \Psi(\lambda_{12}, \mu^0, t^0, z^0, \tau^0) \ = \ \frac{\partial\Phi}{\partial z_a}(\mu^0, t^0, z^0, \tau^0)\, \Psi(\lambda_{12}, \mu^0, t^0, z^0, \tau^0), \ a = 1, \dots, n,$$
$$H_0(z^0, \tau^0)\Psi(\lambda_{12}, \mu^0, t^0, z^0, \tau^0) \ = \ \frac{\partial\Phi}{\partial\tau}(\mu^0, t^0, z^0, \tau^0)\, \Psi(\lambda_{12}, \mu^0, t^0, z^0, \tau^0).$$

Thus, $\Psi(\lambda_{12}, \mu^0, t^0, z^0, \tau^0)$ is a meromorphic eigenfunction of the KZB operators with the eigenvalues given by the partial derivatives of the master function with respect to the parameters z, τ.

Proof. In [FV1], integral representations for solutions of the KZB equations

$$\kappa\partial_{z_a}\psi \ = \ H_a(z, \tau)\psi, \qquad a = 1, \dots, n,$$
$$\kappa\partial_\tau\psi \ = \ H_0(z, \tau)\psi,$$

are constructed starting from horizontal sections of a suitable local system, see [FV1, Proposition 5]. An example of a horizontal section is given by the function

$$e^{\pi i\mu\lambda_{12}}\, \Phi(\mu, t, z, \tau). \tag{5.5}$$

As explained in [RV], integral representations of solutions of the Knizhnik Zamolodchikov type equations can be used to construct common eigenvectors of the commuting systems of operators staying in the right-hand sides of the equations, by applying the stationary phase method to the integral, when $\kappa \to 0$, see [FV1, Section 7]. Applying this procedure to the horizontal section in (5.5), one obtains the eigenfunction $\Psi(\lambda_{12}, \mu, t, z, \tau)$ of Theorem 5.1, see in [FV1] a formula for $\Psi(\lambda_{12}, \mu, t, z, \tau)$. A formula for $\Psi(\lambda_{12}, \mu, t, z, \tau)$ in the special case $V = \otimes_{s=1}^n V_1$ of the tensor product of two-dimensional irreducible $\mathfrak{sl}_2$-modules is given in Section 9. □

5.3. **Fundamental differential operator.** Let $\Psi(\lambda_{12})$ be a $V[0]$-valued eigenfunction of $S_2(x)$,

$$S_2(x)\Psi(\lambda_{12}) = B_2(x)\Psi(\lambda_{12}), \tag{5.6}$$

where $B_2(x)$ is a scalar function of x. We assign to Ψ a scalar differential operator with respect to the variable x,

$$\mathcal{D}_\Psi = \partial_x^2 + B_2(x), \tag{5.7}$$

called the *fundamental differential operator* of the eigenfunction Ψ.

Lemma 5.2. The fundamental differential operator $\mathcal{D}_\Psi$ is doubly periodic,

$$\mathcal{D}_\Psi(x+\tau) = \mathcal{D}_\Psi(x+1) = \mathcal{D}_\Psi(x). \tag{5.8}$$

Moreover, if $s(\mathcal{D}_\Psi)$ is the image of $\mathcal{D}_\Psi$ under the Weyl involution, then $\mathcal{D}_{s(\Psi)} = \mathcal{D}_\Psi$.

Proof. The lemma is a corollary of Theorem 4.1 and Lemma 4.11. □

In particular, let (μ, t, z, τ) be a solution of the Bethe ansatz equations (5.4) (we will omit the index ^{0}). Let $\Psi(\lambda_{12}, \mu, t, z)$ be the corresponding eigenfunction of the KZB operators, see Theorem 5.1. By Theorem 4.9 the coefficient $S_2(x)$ is a linear combination of the KZB operators. Hence $\Psi(\lambda_{12}, \mu, t, z)$ is an eigenfunction of S_2. Let

$$\mathcal{D}_{(\mu,t,z,\tau)} = \partial_x^2 + B_2(x, \mu, t, z, \tau) \tag{5.9}$$

be the fundamental differential operator of $\Psi(\lambda_{12}, \mu, t, z)$. This operator will be also called the *fundamental differential operator* of the solution (μ, t, z, τ).

Remark. In the non-dynamical setting the fundamental differential operator of a solution of the Bethe ansatz equations was introduced in [ScV, MV1].

We will give a formula for $\mathcal{D}_{(\mu,t,z,\tau)}$. Let

$$y(x) = \prod_{i=1}^{m} \theta(x - t_i, \tau), \qquad u(x) = e^{\pi i \mu x} y(x) \prod_{s=1}^{n} \theta(x - z_s, \tau)^{-m_s/2}. \tag{5.10}$$

Theorem 5.3. We have

$$B_2(x, \mu, t, z, \tau) = -(\ln u)'' - ((\ln u)')^2, \tag{5.11}$$

where the function $u(x)$ is defined in (5.10). In other words,

$$\mathcal{D}_{(\mu,t,z,\tau)} = \big(\partial_x + (\ln u)'\big)\big(\partial_x - (\ln u)'\big). \tag{5.12}$$

Proof. First, we calculate $B_2(x, \mu, t, z, \tau)$.

Lemma 5.4. We have

$$4\pi i \frac{\partial}{\partial \tau}(\ln \theta(t - z, \tau)) = \eta(t - z, \tau) - \eta(0).$$

Proof. Recall that we have defined

$$\theta(u, \tau) = \theta_1(u, \tau)/\theta_1'(0, \tau),$$

where the first Jacobi theta function $\theta_1(u, \tau)$ obeys the heat equation

$$4\pi i \frac{\partial}{\partial \tau} \theta_1(u, \tau) = \theta_1''(u, \tau).$$

Since $\theta_1'(u, \tau)$ is holomorphic in u and τ, we have

$$4\pi i \frac{\partial}{\partial \tau} \theta_1'(u, \tau) = 4\pi i \frac{\partial}{\partial u}\frac{\partial}{\partial \tau} \theta_1(u, \tau) = \theta_1'''(u, \tau).$$

Thus

$$4\pi i\frac{\partial}{\partial\tau}(\ln\theta(t-z,\tau))=\frac{\theta_1''(t-z,\tau)}{\theta_1(t-z,\tau)}-\frac{\theta_1'''(0,\tau)}{\theta_1'(0,\tau)}=\eta(t-z,\tau)-\eta(0).$$

□

The eigenvalue of the operator $-4\pi i H_0(z,\tau)$ on the eigenfunction $\Psi(\lambda_{12},t,\mu,z,\tau)$ equals

$$\begin{aligned}
&-2(\pi i)^2\mu^2-4\pi i\frac{\partial}{\partial\tau}\Big[2\sum_{i<j}\ln\theta(t_i-t_j)-\sum_{i,s}m_s\ln\theta(t_i-z_s)+\sum_{s<r}\frac{m_sm_r}{2}\ln\theta(z_s-z_r)\Big]\\
&=-2(\pi i)^2\mu^2-2\sum_{i<j}(\eta(t_i-t_j)-\eta(0))+\sum_{i,s}m_s(\eta(t_i-z_s)-\eta(0))\\
&-\sum_{s<r}\frac{m_sm_r}{2}(\eta(z_s-z_r)-\eta(0))=-2(\pi i)^2\mu^2-2\sum_{i<j}\eta(t_i-t_j)+\sum_{i,s}m_s\eta(t_i-z_s)\\
&-\sum_{s<r}\frac{m_sm_r}{2}\eta(z_s-z_r)+\big(m(m-1)-m\sum_s m_s+\frac12\sum_{s<r}m_sm_r\big)\eta(0)\\
&=-2(\pi i)^2\mu^2-2\sum_{i<j}\eta(t_i-t_j)+\sum_{i,s}m_s\eta(t_i-z_s)-\sum_{s<r}\frac{m_sm_r}{2}\eta(z_s-z_r)\\
&-\big(m(m+1)-\frac12\sum_{s<r}m_sm_r\big)\eta(0).
\end{aligned}$$

For $s=1,\dots,n$, the operator $H_s(z,\tau)$ has eigenvalue

$$\frac{\partial\Phi}{\partial z_s}=-\pi i m_s\mu-\sum_{i=1}^m m_s\rho(z_s-t_i)+\sum_{r\neq s}\frac{m_sm_r}{2}\rho(z_s-z_r),$$

so we have

$$\begin{aligned}
B_2(x,\mu,t,z,\tau)&=-(\pi i)^2\mu^2-\sum_{i<j}\eta(t_i-t_j)+\sum_{i,s}\frac{m_s}{2}\eta(t_i-z_s)-\sum_{s<r}\frac{m_sm_r}{4}\eta(z_s-z_r)\\
&-\frac14\Big(2m(m+1)-\sum_{s<r}m_sm_r\Big)\eta(0)+\pi i\mu\sum_{s=1}^n m_s\rho(x-z_s)+\sum_{s,i}m_s\rho(z_s-t_i)\rho(x-z_s)\\
&\quad-\sum_{s\neq r}\frac{m_sm_r}{2}\rho(z_s-z_r)\rho(x-z_s)+\frac14\sum_s m_s(m_s+2)\rho'(x-z_s).
\end{aligned}$$

Next we calculate the right-hand side in (5.11), namely, the function $R(x,\mu,t,z,\tau)=-(\ln u)''-((\ln u)')^2$. We have

$$\begin{aligned}
(\ln u)'&=\pi i\mu+\sum\nolimits_{i=1}^m\rho(x-t_i)-\frac12\sum\nolimits_{s=1}^m m_s\rho(x-z_s),\\
-(\ln u)''&=-\sum\nolimits_i\rho'(x-t_i)+\frac12\sum\nolimits_s m_s\rho'(x-z_s),
\end{aligned}$$

$$-((\ln u)')^2 = -(\pi i)^2\mu^2 - \sum\nolimits_i \rho(x-t_i)^2 - \frac{1}{4}\sum\nolimits_s m_s^2\rho(x-z_s)^2$$
$$+ \pi i\mu\Big(\sum\nolimits_s m_s\rho(x-z_s) - 2\sum\nolimits_i \rho(x-t_i)\Big) - \sum\nolimits_{i\neq j}\rho(x-t_i)\rho(x-t_j)$$
$$- \frac{1}{4}\sum\nolimits_{s\neq r} m_s m_r\rho(x-z_s)\rho(x-z_r) + \sum\nolimits_{i,s} m_s\rho(x-t_i)\rho(x-z_s),$$

$$R(x,\mu,t,z,\tau) = -\frac{d}{dx}\ln' u - (\ln' u)^2 = -(\pi i)^2\mu^2 - \sum_i \eta(x-t_i) - \frac{1}{4}\sum_s m_s^2\eta(x-z_s)$$
$$+ \frac{1}{4}\sum\nolimits_s m_s(m_s+2)\rho'(x-z_s) + \pi i\mu\Big(\sum\nolimits_s m_s\rho(x-z_s) - 2\sum\nolimits_i \rho(x-t_i)\Big)$$
$$- \sum\nolimits_{i\neq j}\rho(x-t_i)\rho(x-t_j) - \frac{1}{4}\sum\nolimits_{s\neq r} m_s m_r\rho(x-z_s)\rho(x-z_r)$$
$$+ \sum\nolimits_{i,s} m_s\rho(x-t_i)\rho(x-z_s).$$

The function $R(x,t,\mu,z,\tau)$ is doubly periodic in x, since

$$R(x+\tau,t,\mu,z,\tau) - R(x,t,\mu,z,\tau)$$
$$= \sum\nolimits_i \big[4\pi i\rho(x-t_i) - (2\pi i)^2\big] + \sum\nolimits_s m_s^2\big[\pi i\rho(x-z_s) - (\pi i)^2\big]$$
$$- \pi i\mu\Big(\sum\nolimits_s 2\pi i m_s - 2\sum\nolimits_i 2\pi i\Big) + \sum\nolimits_{i\neq j}\big[2\pi i\big(\rho(x-t_i)+\rho(x-t_j)\big) - (2\pi i)^2\big]$$
$$+ \sum\nolimits_{s\neq r} m_s m_r\Big[\frac{\pi i}{2}\big(\rho(x-z_s)+\rho(x-z_r)\big) - (\pi i)^2\Big]$$
$$+ \sum\nolimits_{i,s} m_s\big[(2\pi i)^2 - 2\pi i\big(\rho(x-t_i)+\rho(x-z_s)\big)\big]$$
$$= (\pi i)^2\Big[-4m - \sum\nolimits_s m_s^2 - \mu(2\sum\nolimits_s m_s - 4m) - 4m(m-1) - \sum\nolimits_{s\neq r} m_s m_r + 8m^2\Big]$$
$$+ \pi i\sum\nolimits_i \big(4 + 4(m-1) - 2\sum\nolimits_s m_s\big)\rho(x-t_i)$$
$$+ \pi i\sum\nolimits_s \big(m_s^2 + m_s\sum\nolimits_{r\neq s} m_r - 2mm_s\big)\rho(x-z_s) = 0.$$

Consider the Laurent expansion of $R(x)$ at each pole. Clearly the coefficient in $R(x)$ of $(x-t_i)^{-2}$ is zero. The coefficient in $R(x)$ of $(x-t_i)^{-1}$ equals

$$-2\pi i\mu - 2\sum\nolimits_{j\neq i}\rho(t_i-t_j) + \sum\nolimits_s m_s\rho(t_i-z_s) = 0.$$

Hence $R(x)$ is regular at $x=t_i$ for all i.

The coefficient in $R(x)$ of $(x-z_s)^{-2}$ equals $-\frac{1}{4}m_s(m_s+2)$. The coefficient in $R(x)$ of $(x-z_i)^{-1}$ equals

$$\pi i\mu m_s - \frac{1}{2}\sum\nolimits_{r\neq s} m_s m_r\rho(z_s-z_r) + \sum\nolimits_i m_s\rho(z_s-t_i).$$

These calculations show that the function $B_2(x,\mu,t,z,\tau)$ has the same set of poles in x and the same Laurent tails at each pole in x as the function $R(x,\mu,t,z,\tau)$. Since both functions are doubly periodic in x, we may conclude that $B_2(x,\mu,t,z,\tau) - R(x,\mu,t,z,\tau)$ is constant in x. We need to show that this difference is zero.

We will use the following notion of the constant term of a meromorphic doubly periodic function $F(x)$, regular in the complement to the union of the $\mathbb{Z}+\tau\mathbb{Z}$-orbits of the points $z_1,\dots,z_n$. Namely, let

$$F(x)=\sum\nolimits_{j=1}^{\infty}\frac{c_{i,j}}{(x-z_i)^j}+\mathcal{O}(1),\qquad i=1,\dots,n,$$

be the Laurent expansion of $F(x)$ at $x=z_i$. Then

$$F(x)=\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=1}^{N}\frac{(-1)^{j-1}}{(j-1)!}\,\rho^{(j-1)}(x-z_i,\tau)c_{i,j}\ +\ c_0,$$

where the number $c_0\in\mathbb{C}$ will be called the *constant term* of $F(x)$, cf. Theorem 4.6.

We need to show that the constant term of $B_2(x,\mu,t,z,\tau)$:

$$-(\pi i)^2\mu^2-\sum_{i<j}\eta(t_i-t_j)+\sum_{i,s}\frac{m_s}{2}\eta(t_i-z_s)-\sum_{s<r}\frac{m_sm_r}{4}\eta(z_s-z_r)$$
$$-\frac{1}{4}\Big(2m(m+1)-\sum_{s<r}m_sm_r\Big)\eta(0)$$

equals the constant term of $R_2(x,\mu,t,z,\tau)$:

$$-(\ln u)''-((\ln u)')^2-\sum\nolimits_s\frac{m_s(m_s+2)}{4}\rho'(x-z_s)$$
$$-\sum\nolimits_s\Big[\pi i\mu m_s-\sum\nolimits_{r\neq s}\frac{m_sm_r}{2}\rho(z_s-z_r)+\sum\nolimits_i m_s\rho(z_s-t_i)\Big]\rho(x-z_s)$$

$$=-(\pi i)^2\mu^2-\sum\nolimits_i\eta(x-t_i)-\frac{1}{4}\sum\nolimits_s m_s^2\eta(x-z_s)$$
$$+\frac{1}{4}\sum\nolimits_s m_s(m_s+2)\rho'(x-z_s)-2\pi i\mu\sum\nolimits_i\rho(x-t_i)$$

$$+\sum\nolimits_s\Big[\pi i\mu m_s-\frac{1}{4}\sum\nolimits_{r\neq s}m_sm_r\rho(x-z_r)+\sum\nolimits_i m_s\rho(x-t_i)\Big]\rho(x-z_s)$$
$$-\sum\nolimits_{i\neq j}\rho(x-t_i)\rho(x-t_j)-\sum\nolimits_s\frac{m_s(m_s+2)}{4}\rho'(x-z_s)$$

$$-\sum\nolimits_s\Big[\pi i\mu m_s-\sum\nolimits_{r\neq s}\frac{m_sm_r}{2}\rho(z_s-z_r)+\sum\nolimits_i m_s\rho(z_s-t_i)\Big]\rho(x-z_s)$$
$$=-(\pi i)^2\mu^2-\sum\nolimits_i\eta(x-t_i)-\frac{1}{4}\sum\nolimits_s m_s^2\eta(x-z_s)$$

$$-2\pi i\mu\sum\nolimits_i\rho(x-t_i)-\sum\nolimits_{i\neq j}\rho(x-t_i)\rho(x-t_j)$$
$$+\sum\nolimits_{s\neq r}\frac{m_sm_r}{4}\big(2\rho(z_s-z_r)-\rho(x-z_r)\big)\rho(x-z_s)$$

$$+\sum\nolimits_{i,s}m_s\big(\rho(x-t_i)-\rho(z_s-t_i)\big)\rho(x-z_s)$$

$$\begin{aligned} = \ & -(\pi i)^2\mu^2 - \sum\nolimits_i \eta(x-t_i) - \frac{1}{4}\sum\nolimits_s m_s^2 \eta(x-z_s) \\ & + \sum\nolimits_{s,i} m_s \big[\rho(t_i-x)\rho(t_i-z_s) + \rho(x-t_i)\rho(x-z_s) + \rho(z_s-x)\rho(z_s-t_i)\big] \\ & - 2\sum\nolimits_{i<j} \big[\rho(t_i-t_j)\rho(t_i-x) + \rho(x-t_j)\rho(x-t_i) + \rho(t_j-t_i)\rho(t_j-x)\big] \\ & - \sum\nolimits_{s<r} \frac{m_s m_r}{2} \big[\rho(z_s-z_r)\rho(z_s-x) + \rho(x-z_r)\rho(x-z_s) + \rho(z_r-z_s)\rho(z_r-x)\big]. \end{aligned} \tag{5.13}$$

In the calculation of the expression in (5.13) we use the Bethe ansatz equation to substitute

$$-2\pi i\mu \sum\nolimits_i \rho(x-t_i) = \Big[2\sum\nolimits_{j\neq i}\rho(t_i-t_j) - \sum\nolimits_s m_s \rho(t_i-z_s)\Big]\sum\nolimits_i \rho(x-t_i).$$

According to the above calculations the constant term of $B_2(x,t,\mu,z,\tau)$–$R(x,t,\mu,z,\tau)$ equals the following expression:

$$\begin{aligned} f(t,z) = & -\sum\nolimits_{i<j}\eta(t_i-t_j) + \sum\nolimits_{i,s}\frac{m_s}{2}\eta(t_i-z_s) - \sum\nolimits_{s<r}\frac{m_s m_r}{4}\eta(z_s-z_r) \\ & - \frac{1}{4}\Big(2m(m+1) + \sum\nolimits_{s<r} m_s m_r\Big)\eta(0) + \sum\nolimits_i \eta(x-t_i) + \frac{1}{4}\sum\nolimits_s m_s^2 \eta(x-z_s) \\ & - \sum\nolimits_{s,i} m_s \big[\rho(t_i-x)\rho(t_i-z_s) + \rho(x-t_i)\rho(x-z_s) + \rho(z_s-x)\rho(z_s-t_i)\big] \\ & + 2\sum\nolimits_{i<j} \big[\rho(t_i-t_j)\rho(t_i-x) + \rho(x-t_j)\rho(x-t_i) + \rho(t_j-t_i)\rho(t_j-x)\big] \\ & + \sum\nolimits_{s<r} \frac{m_s m_r}{2} \big[\rho(z_s-z_r)\rho(z_s-x) + \rho(x-z_r)\rho(x-z_s) + \rho(z_r-z_s)\rho(z_r-x)\big]. \end{aligned}$$

Lemma 5.5. The function $f(t,z)$ is regular and doubly periodic in each of the variables $t_1, \dots, t_m, z_1, \dots, z_n$ and hence is a constant in t and z.

Proof. The regularity is easily checked by calculating the residues. The periodicity is checked as follows:

$$\begin{aligned} f(\dots,t_i+\tau,\dots) - f(\dots,t_i,\dots) = & \sum\nolimits_{j\neq i}\big[4\pi i\rho(t_i-t_j) - (2\pi i)^2\big] \\ & + \sum\nolimits_s \frac{m_s}{2}\big[(2\pi i)^2 - 4\pi i\rho(t_i-z_s)\big] + \big[(2\pi i)^2 - 4\pi i\rho(t_i-x)\big] \\ & + \sum\nolimits_s m_s\big[2\pi i\big(\rho(t_i-x)+\rho(t_i-z_s)\big) - (2\pi i)^2\big] \\ & + 2\sum\nolimits_{j\neq i}\big[(2\pi i)^2 - 2\pi i\big(\rho(t_i-t_j)+\rho(t_i-x)\big)\big] \\ = & \ (\pi i)^2\big[-4(m-1) + 2\sum\nolimits_s m_s + 4 - 4\sum\nolimits_s m_s + 8(m-1)\big] \\ & + 4\pi i\sum\nolimits_{j\neq i}\big(1-1\big)\rho(t_i-t_j) + \pi i\sum\nolimits_s\big(-2m_s+2m_s\big)\rho(t_i-z_s) \\ & + \pi i\big(-4+2\sum\nolimits_s m_s - 4(m-1)\big)\rho(t_i-x) = 0; \end{aligned}$$

$$
\begin{aligned}
f(\dots, z_s+\tau, \dots) - f(\dots, z_s, \dots) &= \sum\nolimits_i \frac{m_s}{2}\big[(2\pi i)^2 - 4\pi i \rho(z_s - t_i)\big] \\
&+ \sum\nolimits_{r\neq s} \frac{m_s m_r}{4}\big[4\pi i\rho(z_s - z_r) - (2\pi i)^2\big] + \frac{1}{4} m_s^2 \big[(2\pi i)^2 - 4\pi i \rho(z_s - x)\big] \\
&+ \sum\nolimits_i m_s \big[2\pi i\big(\rho(z_s - x) + \rho(z_s - t_i)\big) - (2\pi i)^2\big] \\
&+ \frac{1}{2}\sum\nolimits_{r\neq s} m_s m_r \big[(2\pi i)^2 - 2\pi i\big(\rho(z_s - z_r) + \rho(z_s - x)\big)\big] \\
= (\pi i)^2 \big[2mm_s - m_s \sum\nolimits_{r\neq s} m_r &+ m_s^2 - 4mm_s + 2m_s \sum\nolimits_{r\neq s} m_r\big] \\
&+ \pi i \sum\nolimits_i \big(-2m_s + 2m_s\big)\rho(z_s - t_i) + \pi i \sum\nolimits_{r\neq s}\big(m_s m_r - m_s m_r\big)\rho(z_s - z_r) \\
&+ \pi i\big(-m_s^2 + 2mm_s - \sum\nolimits_{r\neq s} m_s m_r\big)\rho(z_s - x) = 0.
\end{aligned}
$$

□

By Lemma 5.5 the function $f(t, z)$ is constant in t and z, while its value at $t = 0$, $z = 0$ is

$$
\begin{aligned}
\eta(0)\Big[-\frac{m(m-1)}{2} + m^2 - \sum\nolimits_{s<r} \frac{m_s m_r}{4} - \frac{m(m+1)}{2} \\
+ \frac{1}{4}\sum\nolimits_{s<r} m_s m_r + m + \frac{1}{4}\sum\nolimits_s m_s^2 - 2m^2 + m(m-1) + \frac{1}{2}\sum\nolimits_{s<r} m_s m_r\Big] = 0,
\end{aligned}
$$

see Lemma 2.3. Hence $f(t, z)$ equals zero and Theorem 5.3 is proved. □

6. Special case $V = (\mathbb{C}^2)^{\otimes 2m}$

In the remainder of this paper we consider the situation of Section 5. In addition to the assumptions of Section 5 we will always assume that

$$m_1 = \dots = m_n = 1, \quad \text{and so} \quad n = 2m;$$

that is, from now on we will study the dynamical elliptic Bethe algebra $\mathcal{B}^V(z_1, \dots, z_{2m}, \tau)$ on the zero-weight subspace of the tensor product

$$V = (\mathbb{C}^2)^{\otimes 2m}$$

of two-dimensional irreducible $\mathfrak{sl}_2$-modules.

In this case the elliptic master function (5.3) becomes

$$
\begin{aligned}
\Phi(t, \mu, z, \tau) = \frac{\pi i}{2}\mu^2\tau + 2\pi i\mu\Big(\sum\nolimits_{i=1}^m t_i - \sum\nolimits_{s=1}^{2m}\frac{1}{2}z_s\Big) + 2\sum_{1\leqslant i<j\leqslant m} \ln\theta(t_i - t_j, \tau) \\
- \sum\nolimits_{i=1}^m \sum\nolimits_{s=1}^{2m} \ln\theta(t_i - z_s, \tau) + \sum_{1\leqslant s<r\leqslant 2m} \frac{1}{2}\ln\theta(z_s - z_r, \tau), \qquad (6.1)
\end{aligned}
$$

the Bethe ansatz equations (5.4) become

$$2\pi i\mu + 2\sum\nolimits_{k,\, k\neq j}\rho(t_j - t_k, \tau) - \sum\nolimits_{s=1}^{2m}\rho(t_j - z_s, \tau) = 0, \qquad j = 1, \dots, m. \quad (6.2)$$

7. Theta-polynomials

7.1. **Definitions.** Fix $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$.

Definition 7.1. A *theta-polynomial of degree* m is a function of the form

$$f(x) = ce^{2\pi i\mu x}\prod_{j=1}^{m}\theta(x-t_j,\tau) \tag{7.1}$$

where $c, \mu, t_1, \dots, t_m \in \mathbb{C}$. We have

$$\begin{aligned} f(x+1) &= e^{2\pi i\mu}(-1)^m f(x), \\ f(x+\tau) &= e^{2\pi i\mu\tau}(-1)^m e^{-\pi i m\tau - 2\pi i m x + 2\pi i\sum_{j=1}^m t_j} f(x). \end{aligned} \tag{7.2}$$

The *first and second multipliers* of the theta-polynomial are the numbers

$$A = e^{2\pi i\mu}, \qquad B = e^{2\pi i\mu\tau + 2\pi i\sum_{j=1}^m t_j}. \tag{7.3}$$

Definition 7.2. Given $A, B \in \mathbb{C}^\times$, an entire function $f(x)$ is called a *theta-polynomial of degree* m with multipliers A, B if

$$f(x+1) = A(-1)^m f(x), \qquad f(x+\tau) = B(-1)^m e^{-\pi i m\tau - 2\pi i m x} f(x). \tag{7.4}$$

Clearly if $f(x)$ satisfies Definition 7.1 then it satisfies Definition 7.2.

Lemma 7.3. Let $f(x)$ satisfies Definition 7.2 with multipliers A, B. Let $\mu \in \mathbb{C}$ be such that $A = e^{2\pi i\mu}$. Then there exist $t_1, \dots, t_m \in \mathbb{C}$ such that

$$f(x) = ce^{2\pi i\mu x}\prod_{j=1}^{m}\theta(x-t_j,\tau). \tag{7.5}$$

Proof. The function $g(x) = e^{-2\pi i\mu x} f(x)$ has the transformation properties:

$$g(x+1) = (-1)^m g(x), \qquad g(x+\tau) = e^{-2\pi i\mu\tau} B(-1)^m e^{-\pi i m\tau - 2\pi i m x} g(x).$$

Let $s_1, \dots, s_m \in \mathbb{C}$ be any numbers such that

$$e^{2\pi i\sum_{j=1}^m s_j} = e^{-2\pi i\mu\tau} B.$$

Then the product $h(x) = \prod_{j=1}^m \theta(x - s_j, \tau)$ has the transformation properties:

$$\begin{aligned} h(x+1) &= (-1)^m h(x), \\ h(x+\tau) &= (-1)^m e^{-\pi i m\tau - 2\pi i m x + 2\pi i\sum_{j=1}^m s_j} h(x) = e^{-2\pi i\mu\tau} B(-1)^m e^{-\pi i m\tau - 2\pi i m x} h(x). \end{aligned}$$

The function g/h is doubly periodic with m poles at the points $[s_1], \dots, [s_m]$ on the elliptic curve $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and m zeros at some points $[t_1], \dots, [t_m]$ on the elliptic curve $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. By Theorem 20.14 in [WW] we have $\sum_{j=1}^m [t_j] = \sum_{j=1}^m [s_j]$ on $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. Choose representatives $t_1, \dots, t_m \in \mathbb{C}$ of $[t_1], \dots, [t_m]$ such that $\sum_{j=1}^m s_j = \sum_{j=1}^m t_j$. Define $u(x) = \prod_{j=1}^m \theta(x - t_j, \tau)$. Then

$$\begin{aligned} u(x+1) &= (-1)^m u(x), \\ u(x+\tau) &= (-1)^m e^{-\pi i m\tau - 2\pi i m x + 2\pi i\sum_{j=1}^m t_j} u(x) = (-1)^m e^{-\pi i m\tau - 2\pi i m x + 2\pi i\sum_{j=1}^m s_j} u(x). \end{aligned}$$

Hence

$$\frac{g(x)}{h(x)}\frac{h(x)}{u(x)} = \frac{g(x)}{u(x)}$$

is an entire doubly periodic function. Hence it is some constant $c \in \mathbb{C}$ and we obtain (7.5). ∎

Notice that the numbers $c, \mu, t_1, \dots, t_m$ are not unique.

Lemma 7.4. If

$$ce^{2\pi i\mu x}\prod_{j=1}^{m}\theta(x-t_j,\tau) \quad \text{and} \quad c'e^{2\pi i\mu' x}\prod_{j=1}^{m}\theta(x-t'_j,\tau) \tag{7.6}$$

are presentations of the same function, then

(i) after a suitable permutation of $t_1, \dots, t_m$ we have $t'_j = t_j - k_j - l_j\tau$ for $j = 1, \dots, m$ and some $k_j, l_j \in \mathbb{Z}$;

(ii) $\mu = \mu' - \sum_{j=1}^{m} l_j$;

(iii)

$$c = c'(-1)^{\sum_{j=1}^{m}(k_j+l_j)}e^{-\pi i\sum_{j=1}^{m}k_j^2\tau+2\pi i\sum_{j=1}^{m}k_jt_j}. \tag{7.7}$$

Conversely, for any c, c', μ, μ', $t_1, \dots, t_m$, $t'_1, \dots, t'_m$ satisfying conditions (i-iii) the two functions in (7.6) are equal.

Proof. Statement (i) is obvious. The second presentation in (7.6) takes the form

$$c'e^{2\pi i\mu' x}(-1)^{\sum_{j=1}^{m}(k_j+l_j)}e^{-\pi i\sum_{j=1}^{m}k_j^2\tau+2\pi i\sum_{j=1}^{m}k_jt_j}e^{-2\pi i\sum_{j=1}^{m}k_jx}\prod_{j=1}^{m}\theta(x-t_j,\tau).$$

Comparing the two presentations we obtain the lemma. □

7.2. **Space** $T_{m,A,B}$**.** Given $m \in \mathbb{Z}_{>0}$ and $A, B \in \mathbb{C}^\times$, denote by $T_{m,A,B}$ the vector space of theta-polynomials of degree m with multipliers A, B.

Lemma 7.5. The vector space $T_{m,A,B}$ has dimension m.

Proof. Let $\nu \in \mathbb{C}$ be such that $e^{2\pi i\nu} = A(-1)^m$. Then for any $f \in T_{n,A,B}$, the function $g(x) = e^{-2\pi i\nu x}f(x)$ has the properties:

$$g(x+1) = g(x), \qquad g(x+\tau) = e^{-2\pi i\nu\tau}B(-1)^m e^{-\pi i m\tau-2\pi i m x}g(x).$$

Let $g(x) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi i k x}$ be the Fourier expansion and $q = e^{2\pi i\tau}$. Then

$$\sum_{k\in\mathbb{Z}} a_k q^k e^{2\pi i k x} = e^{-2\pi i\nu\tau}B(-1)^m e^{-2\pi i m x}q^{-m/2}\sum_{k\in\mathbb{Z}} a_k e^{2\pi i k x}$$

or

$$a_{k+m} = e^{2\pi i\nu\tau}B^{-1}(-1)^m q^{k+m/2}a_k. \tag{7.8}$$

Arbitrary choice of coefficients $a_1, \dots, a_m$ determines $g(x)$ uniquely. The map sending a theta-polynomial to the vector $(a_1, \dots, a_m)$ identifies $T_{m,A,B}$ with $\mathbb{C}^m$. □

For $\nu \in \mathbb{C}$ the map

$$L_\nu : T_{m,A,B} \to T_{m,e^{2\pi i\nu}A,\, e^{2\pi i\nu\tau}B}, \qquad f(x) \mapsto e^{2\pi i\nu x}f(x) \tag{7.9}$$

is an isomorphism of vector spaces.

7.3. **Spaces $T_{m,A}$, T_m.** For $A \in \mathbb{C}^\times$ denote by $T_{m,A}$ the space of theta-polynomials of degree m with first multiplier A and by T_m the space of all theta-polynomials of degree m. Fix $\mu \in \mathbb{C}$ such that $e^{2\pi i\mu} = A$. Then all elements of $T_{m,A}$ have the form

$$f(x) = e^{2\pi i\mu x}\prod_{j=1}^{m}\theta(x - t_j, \tau)$$

with arbitrary $t_1, \dots, t_m \in \mathbb{C}$, see Lemma 7.3. We have

$$T_{m,A} = \cup_{B\in\mathbb{C}^\times} T_{m,A,B}, \quad T_m = \cup_{A,B\in\mathbb{C}^\times} T_{m,A,B} = \cup_{A\in\mathbb{C}^\times} T_{m,A}.$$

The maps L_ν, $\nu \in \mathbb{C}$, define an action of $\mathbb{C}$ on T_m.

7.4. **Projectivization.** Denote by $P_{m,A,B}$ the space of nonzero theta-polynomials of degree m with multipliers A, B identified up to multiplication by a nonzero number. Thus $P_{m,A,B}$ is $m-1$-dimensional projective space.

Denote by $P_{m,A}$ the space of nonzero theta-polynomials of degree m with first multiplier A identified up to multiplication by a nonzero number and by P_m the space of nonzero theta-polynomials of degree n identified up to multiplication by a nonzero number. We have

$$P_m = \cup_{A\in\mathbb{C}^\times} P_{m,A} = \cup_{A,B\in\mathbb{C}^\times} P_{m,A,B}.$$

The maps L_ν, $\nu \in \mathbb{C}$, define an action of the group $\mathbb{C}$ on P_m. In particular the maps L_k, $k \in \mathbb{Z}$, define an action of $\mathbb{Z}$ on each $P_{m,A}$.

The group $\mathbb{C}$ acts on $P_m \times P_m$ by the maps $L_\nu \times L_\nu$. This action preserves

$$\cup_{A_1,A_2\in\mathbb{C}^\times,\, A_1\neq A_2} P_{m,A_1} \times P_{m,A_2}.$$

One of the spaces that we are interested in this paper is the space

$$(\cup_{A_1,A_2\in\mathbb{C}^\times,\, A_1\neq A_2} P_{m,A_1} \times P_{m,A_2})/\mathbb{C} \cong (P_{m,1} \times (P_m - P_{m,1}))/\mathbb{Z} \tag{7.10}$$

of dimension $2m + 1$.

7.5. **Analytic involution.** Define the *analytic involution*

$$P_m \times P_m \to P_m \times P_m, \qquad (f, g) \mapsto (g, f). \tag{7.11}$$

The analytic involution commute with the diagonal action of $\mathbb{C}$ on $P_m \times P_m$ by the maps $\Lambda_\nu \times L_\nu$, $\nu \in \mathbb{C}$.

The analytic involution on $P_m \times P_m$ descends to the *analytic involution*

$$\iota_{\mathrm{an}} : (P_{m,1} \times (P_m - P_{m,1}))/\mathbb{Z} \to (P_{m,1} \times (P_m - P_{m,1}))/\mathbb{Z} \tag{7.12}$$

as follows. Take a point

$$p \in (P_{m,1} \times P_{m,A^{-1}})/\mathbb{Z} \subset (P_{m,1} \times (P_m - P_{m,1}))/\mathbb{Z}$$

with $A \neq 1$. Choose its representative $(f(x), g(x)) \in P_{m,1} \times P_{m,A^{-1}}$. Let $\mu \in \mathbb{C}$ be such that $e^{2\pi i\mu} = A$. Define $\iota_{\mathrm{an}}(p)$ to be the equivalence class of $(e^{2\pi i\mu x} g(x), e^{2\pi i\mu x} f(x))$ in $(P_{m,1} \times P_{m,A}))/\mathbb{Z}$. We have $\iota_{\mathrm{an}}^2 = \mathrm{Id}$.

8. Wronskian determinant

8.1. **Definition.** For functions $f(x)$, $g(x)$ the determinant

$$\mathrm{Wr}(f,g) = f(x)g'(x) - f'(x)g(x)$$

is called the *Wronskian determinant.* We have

$$\mathrm{Wr}(hf,hg) = h^2\,\mathrm{Wr}(f,g) \tag{8.1}$$

for any function h.

Lemma 8.1. If $f \in T_{m_1,A_1,B_1}$, $g \in T_{m_2,A_2,B_2}$, then $fg \in T_{m_1+m_2,A_1A_2,B_1B_2}$. □

Lemma 8.2. If $f \in T_{m,A_1,B_1}$, $g \in T_{m,A_2,B_2}$, then

$$\mathrm{Wr}(f,g) = h,$$

where $h \in T_{2m,A_1A_2,B_1B_2}$.

Proof. The function

$$\mathrm{Wr}(f,g)/fg = g'/g - f'/f$$

is doubly periodic. Hence the entire function $\mathrm{Wr}(f,g)$ has the same transformation properties as $fg \in T_{2m,A_1A_2,B_1B_2}$. □

8.2. **Wronskian equation.**

Lemma 8.3. If $f(x), g(x)$ are holomorphic functions, then the meromorphic function $(g/f)' = \mathrm{Wr}(f,g)/f^2$ has zero residue at every pole. □

Theorem 8.4 ([BMV]). Let $f \in T_{m,A_1,B_1}$, $h \in T_{2m,A_1A_2,B_1B_2}$ be nonzero functions such that $(A_1,B_1) \neq (A_2,B_2)$. Let all zeros of $f(x)$ be simple. Let the function h/f^2 have zero residue at every zero of f. Then there exists a unique $g \in T_{m,A_2,B_2}$ such that $\mathrm{Wr}(f,g) = h$.

Proof. The function $m = h/f^2$ satisfies equations

$$m(x+1) = A\,m(x), \qquad m(x+\tau) = B\,m(x),$$

where $A = A_2/A_1$, $B = B_2/B_1$ and hence $(A,B) \neq (1,1)$. Choose $x_0 \in \mathbb{C}$ not a pole of m and define $M(x) = \int_{x_0}^{x} m(u)du$. Then

$$M(x+1) = A\,M(x) + a, \qquad a = \int_{x_0}^{x_0+1} m(u)du,$$

$$M(x+\tau) = B\,M(x) + b, \qquad b = \int_{x_0}^{x_0+\tau} m(u)du.$$

Lemma 8.5. We have

$$a(B-1) = b(A-1). \tag{8.2}$$

Proof. The integral of m over the boundary of the parallelogram with vertices at x_0, $x_0+1, x_0+1+\tau, x_0+\tau$ is zero since m has no residues. On the other hand it equals

$$\int_{x_0}^{x_0+1} m(x)dx + \int_{x_0+1}^{x_0+1+\tau} m(x)dx + \int_{x_0+1+\tau}^{x_0+\tau} m(x)dx + \int_{x_0+\tau}^{x_0} m(x)dx$$
$$= \int_{x_0}^{x_0+1} m(x)dx + A\int_{x_0}^{x_0+\tau} m(x)dx + B\int_{x_0+1}^{x_0} m(x)dx + \int_{x_0+\tau}^{x_0} m(x)dx$$
$$= a + Ab - Ba - b.$$

□

Lemma 8.6. Let $M(x)$ be a function such that

$$M(x+1) = A\,M(x) + a, \qquad M(x+\tau) = B\,M(x) + b,$$

for some $A, B \in \mathbb{C}^\times$, $(A,B) \neq (1,1)$, and $a, b \in \mathbb{C}$. Then there exists a unique $C \in \mathbb{C}$ such that the function $\tilde{M}(x) := M(x) + C$ satisfies the equations

$$\tilde{M}(x+1) = A\,\tilde{M}(x), \qquad \tilde{M}(x+\tau) = B\,\tilde{M}(x).$$

Proof. For any C we have

$$\tilde{M}(x+1) = A\,\tilde{M}(x) + C(1-A) + a, \qquad \tilde{M}(x+\tau) = B\,\tilde{M}(x) + C(1-B) + b.$$

We want C to satisfy the system of equations

$$C(1-A) + a = 0, \qquad C(1-B) + b = 0.$$

If $(A,B) \neq (1,1)$, this system has a solution if and only if equation (8.2) holds. □

The function $g(x) = f(x)\tilde{M}(x)$ is holomorphic since all zeros of f are simple. The function g lies in T_{n,A_2,B_2} and satisfies the equation $\mathrm{Wr}(f,g) = h$. □

8.3. **Points with generic coordinates.** We say that a point $(f,g) \in P_m \times P_m$ has *generic first* (resp. *second*) *coordinate* if f (resp. g) has no common zeros with $\mathrm{Wr}(f,g)$.

Lemma 8.7. Assume that $(f,g) \in P_m \times P_m$ has generic first coordinate, then

(i) every point of the $\mathbb{C}$-orbit of (f,g) has generic first coordinate;
(ii) all zeros of f are simple;
(iii) f has no common zeros with g.

□

Lemma 8.8. The complement in $P_{m,1} \times (P_m - P_{m,1})$ to the subset of points with generic both first and second coordinates is a proper analytic subset of $P_{m,1} \times (P_m - P_{m,1})$.

Proof. Clearly, the complement is an analytic subset of $P_{m,1} \times (P_m - P_{m,1})$. Let us show that it is proper. Indeed, every point f of $P_{m,1}$ has a presentation $f(x) = \prod_{j=1}^m \theta(x - t_j, \tau)$ for some $t_1, \dots, t_m \in \mathbb{C}$ by Lemma 7.3. Every point g of $P_m - P_{m,1}$ lies in some $P_{m,A}$ with $A \neq 1$. If $\mu \in \mathbb{C}$ is such that $e^{2\pi i \mu} = A$, then the every point g of $P_{m,A}$ has a presentation of the form $g = e^{2\pi i \mu x} \prod_{j=1}^m \theta(x - s_j, \tau)$ for some $s_1, \dots, s_m \in \mathbb{C}$ by Lemma 7.3. If $t_1, \dots, t_m, s_1, \dots, s_m$ are all distinct modulo

$\mathbb{Z}+\tau\mathbb{Z}$, then the point $(f,g)\in P_{m,1}\times(P_m-P_{m,1})$ has generic both first and second coordinates. □

Denote

$$\begin{aligned}
\mathrm{Pairs}^1_m &= \{p\in (P_{m,1}\times(P_m-P_{m,1}))/\mathbb{Z}\mid p \text{ has generic first coordinate}\},\\
\mathrm{Pairs}^2_m &= \{p\in (P_{m,1}\times(P_m-P_{m,1}))/\mathbb{Z}\mid p \text{ has generic second coordinate}\},\\
\mathrm{Pairs}_m &= \{p\in (P_{m,1}\times(P_m-P_{m,1}))/\mathbb{Z}\mid p \text{ has generic first and second coordinates}\}.
\end{aligned}$$

8.4. **Functions g/f, $\mathrm{Wr}(f,g)/f^2$.** Consider the meromorphic functions of the form

$$F=g/f,\qquad G=\mathrm{Wr}(f,g)/f^2, \tag{8.3}$$

where $(f,g)\in P_{m,1}\times(P_m-P_{m,1})$. We *will consider such functions up to multiplication by nonzero numbers.* Notice that $G=F'$.

Lemma 8.9. The pairs $(f,g)\in\mathrm{Pairs}^1_m$ are in bijective correspondence with the functions $G=\mathrm{Wr}(f,g)/f^2$, which have m distinct poles of order 2 in a fundamental parallelogram for the lattice $\mathbb{Z}+\tau\mathbb{Z}$ acrting on $\mathbb{C}$.

Proof. Let $(f,g)\in P_{m,1}\times(P_m-P_{m,1})$ have generic first coordinate. A pair in the same $\mathbb{Z}$-orbit is $(e^{2\pi ikx}f,\ e^{2\pi ikx}g)$ for some $k\in\mathbb{Z}$. Both pairs define the same function G by formula (8.1).

Conversely assume that there are two pairs (f,g), $(\hat f,\hat g)\in P_{m,1}\times(P_m-P_{m,1})$ with generic first coordinates such that

$$\mathrm{Wr}(f,g)/f^2=\mathrm{Wr}(\hat f,\hat g)/\hat f^2 \tag{8.4}$$

up to a constant factor. By assumption, the functions f and $\hat f$ have the same zeros. Hence $\hat f=e^{2\pi ikx}f$ up to a constant factor. Then $\mathrm{Wr}(\hat f,\hat g)=e^{4\pi ikx}\,\mathrm{Wr}(f,g)$ up to a constant factor. Then $\mathrm{Wr}(e^{2\pi ikx}f,\hat g)=e^{4\pi ikx}\,\mathrm{Wr}(f,g)$ and $\mathrm{Wr}(f,e^{-2\pi ikx}\hat g)=\mathrm{Wr}(f,g)$. By Theorem 8.4 we have $e^{-2\pi ikx}\hat g=g$ up to a constant factor. The lemma is proved. □

Lemma 8.10. The pairs $(f,g)\in\mathrm{Pairs}^1_m$ are in bijective correspondence with functions $F=g/f$, which have m simple poles in a fundamental parallelogram for the lattice $\mathbb{Z}+\tau\mathbb{Z}$ acting on $\mathbb{C}$.

Proof. Let $(f,g)\in P_{m,1}\times(P_m-P_{m,1})$ have generic first coordinate. A pair in the same $\mathbb{Z}$-orbit is $(e^{2\pi ikx}f,\ e^{2\pi ikx}g)$ for some $k\in\mathbb{Z}$. Both pairs define the same function F.

Conversely assume that there are two points $(f,g),(\hat f,\hat g)\in P_{m,1}\times(P_m-P_{m,1})$ with generic first coordinates such that the corresponding two ratios are equal, $g/f=\hat g/\hat f$. Then their derivatives are also equal, see formula (8.4). Hence the two points lie in the same $\mathbb{Z}$-orbit by Lemma 8.9. The lemma is proved. □

8.5. **Fundamental differential operator.** For $(f,g)\in P_{m,1}\times(P_m-P_{m,1})$ introduce two functions

$$u_1(x)=f/\sqrt{\mathrm{Wr}(f,g)},\qquad u_2(x)=g/\sqrt{\mathrm{Wr}(f,g)}. \tag{8.5}$$

Then

$$\mathrm{Wr}(u_1,u_2)=1, \tag{8.6}$$

see (8.1). Let $\mathcal{D}_{(f,g)}$ be the monic linear differential operator, whose kernel is generated by the functions u_1, u_2. This operator will be called the *fundamental differential operator* of the pair (f, g).

Lemma 8.11. We have

$$\mathcal{D}_{(f,g)} = \Big(\partial_x + (\ln u_1)'\Big)\Big(\partial_x - (\ln u_1)'\Big) = \Big(\partial_x + (\ln u_2)'\Big)\Big(\partial_x - (\ln u_2)'\Big). \qquad (8.7)$$

Proof. The functions u_1, u_2 are linearly independent. Hence $\mathcal{D}_{(f,g)}$ is a second order differential operator. We will prove that $\mathcal{D}_{(f,g)}$ equals

$$\Big(\partial_x + (\ln u_1)'\Big)\Big(\partial_x - (\ln u_1)'\Big) = \partial_x^2 - (\ln u_1)'' - ((\ln u_1)')^2. \qquad (8.8)$$

The remaining equality in (8.7) is proved similarly.

Clearly the function u_1 lies in the kernel of the operator in (8.8). Since the differential operator ∂_x enters the right-hand side in (8.8) with zero coefficient, we conclude that the kernel of the operator in (8.8) is generated by two functions, one of which is u_1 and the other, say u is such that $\mathrm{Wr}(u_1, u) = 1$, but u_2 is such a function. □

We have the following properties of the fundamental differential operator $\mathcal{D}_{(f,g)}$.

Lemma 8.12.

(i) The operator $\mathcal{D}_{(f,g)}$ is doubly periodic,

$$\mathcal{D}_{(f,g)}(x + k + l\tau) = \mathcal{D}_{(f,g)}(x), \qquad k, l \in \mathbb{Z}$$

All singular points of $\mathcal{D}_{(f,g)}$ are regular singular.

(ii) The monodromy group of the kernel of the operator $\mathcal{D}_{(f,g)}$ preserves the direct decomposition $\langle u_1\rangle \oplus \langle u_2\rangle$. If $f \in P_{m,A_1,B_1}$, where $A_1 = 1$, and $g \in P_{m,A_2,B_2}$, then the monodromy operator of the transformation $x \to x+1$ has the matrix $\mathrm{diag}(\sqrt{A_1/A_2}, \sqrt{A_2/A_1})$ and the monodromy operator of the transformation $x \to x + \tau$ has the matrix $\mathrm{diag}(\sqrt{B_1/B_2}, \sqrt{B_2/B_1})$.

(iii) Assume that (f, g) has generic first and second coordinates and all zeros of $\mathrm{Wr}(f, g)$ are simple. Then the singular points of $\mathcal{D}_{(f,g)}$ are the $\mathbb{Z} + \tau\mathbb{Z}$-orbits of the set $\{z_1, \dots, z_{2m}\}$. The local monodromy around every singular point has the matrix $\mathrm{diag}(-1, -1)$.

Proof. The operator $\mathcal{D}_{(f,g)}$ is doubly periodic because the functions u_1, u_2 are doubly periodic. The other statements are clear. □

Lemma 8.13. The fundamental differential operator is well-defined for pairs in Pairs_m^1.

Proof. Clearly the pairs (f, g) and $(e^{2\pi ikx} f, e^{2\pi ikx} g)$ have the same functions u_1, u_2, see formula (8.1). □

Recall the analytic involution

$$\iota_{\mathrm{an}} : (P_{m,1} \times (P_m - P_{m,1}))/\mathbb{Z} \to (P_{m,1} \times (P_m - P_{m,1}))/\mathbb{Z}$$

defined in (7.12).

Theorem 8.14. For any $p \in \mathrm{Pairs}_m$ the points p and $\iota_{\mathrm{an}}(p)$ have the same fundamental differential operators. Conversely, if two points $p, \tilde{p} \in \mathrm{Pairs}_m$ have the same fundamental differential operator, then either $\tilde{p} = p$ or $\iota_{\mathrm{an}}(\tilde{p}) = p$.

Proof. The first statement is clear from the definition of the fundamental differential operator. Let us prove the second statement.

For p and $\tilde{p}$ consider the corresponding functions u_1, u_2 and $\tilde{u}_1, \tilde{u}_2$, which generate the kernels of the corresponding fundamental differential operators.

The transformation $x \to x + 1$ defines a linear map on the space $\langle u_1, u_2 \rangle$ with eigenvectors u_1, u_2, which have distinct eigenvalues. Hence u_1 is proportional to $\tilde{u}_1$ and u_2 is proportional to $\tilde{u}_2$ or u_1 is proportional to $\tilde{u}_2$ and u_2 is proportional to $\tilde{u}_1$.

In the first case, the function u_1^2 is proportional to $\tilde{u}_1^2$ and then $\tilde{p} = p$ by Lemma 8.9. In the second case, the function u_1^2 is proportional to $\tilde{u}_2^2$ and $\iota_{\mathrm{an}}(\tilde{p}) = p$ by Lemma 8.9. □

9. Bethe ansatz equations and pairs of theta-polynomials

9.1. **Bethe ansatz equations and Wronskian equation.** First notice that if $(\mu, t_1, \ldots, t_m, z_1, \ldots, z_{2m})$ is a solution of the Bethe ansatz equations (6.2), then $t_1, \ldots, t_m$ are distinct modulo the lattice $\mathbb{Z} + \tau\mathbb{Z}$ and any t_j is distinct from any z_a modulo the lattice $\mathbb{Z} + \tau\mathbb{Z}$.

Lemma 9.1. Given $\mu, t_1, \ldots, t_m, z_1, \ldots, z_{2m}$ let $t_1, \ldots, t_m$ be distinct numbers modulo the lattice $\mathbb{Z} + \tau\mathbb{Z}$ and let any t_j be distinct from any z_a modulo the lattice $\mathbb{Z} + \tau\mathbb{Z}$. Then the function

$$G(x) = e^{-2\pi i\mu x} \frac{\prod_{a=1}^{2m} \theta(x - z_a, \tau)}{\prod_{j=1}^{m} \theta(x - t_j, \tau)^2} \tag{9.1}$$

has zero residues with respect to x at any point, if and only if the numbers $\mu, t_1, \ldots, t_m$, $z_1, \ldots, z_{2m}$ satisfy the Bethe ansatz equations (6.2).

Proof. We have

$$G(x+1) = e^{-2\pi i\mu} G(x), \qquad G(x+\tau) = e^{-2\pi i\mu\tau} e^{2\pi i(\sum_{s=1}^{2m} z_s - 2\sum_{j=1}^{m} t_j)} G(x).$$

Hence it is enough to check that $G(x)$ has zero residues at the points $x = t_j, j = 1, \ldots, m$. We have $\theta(x) = \theta'(0)x + o(x^2)$ as $x \to 0$, since $\theta(x)$ is odd. As $x - t_j \to 0$, we have

$$G(x, \mu, t, z, \tau) = e^{-2\pi i\mu x} \frac{\prod_{a=1}^{2m} \theta(x - z_a)}{(\theta'(0)(x - t_j) + o((x - t_j)^2)^2 \prod_{k \neq j} \theta(x - t_k)^2}.$$

Hence $\mathrm{Res}_{x=t_j} G(x) = 0$ if and only if the logarithmic derivative of

$$e^{-2\pi i\mu x} \frac{\prod_{a=1}^{2m} \theta(x - z_a)}{\prod_{k \neq j} \theta(x - t_k)^2}$$

at $x = t_j$ equals zero. That is exactly the j-th Bethe ansatz equation. □

Lemma 9.2. Let $(f, g) \in P_{m,1} \times (P_m - P_{m,1})$ have generic first coordinate and $f = \prod_{j=1}^{m} \theta(x - t_j, \tau)$, $\mathrm{Wr}(f, g) = e^{-2\pi i\mu x} \prod_{a=1}^{2m} \theta(x - z_a, \tau)$ for some $\mu, t_1, \ldots, t_m, z_1, \ldots, z_{2m}$. Then $(\mu, t_1, \ldots, t_m, z_1, \ldots, z_{2m})$ is a solution of the Bethe ansatz equations (6.2).

Proof. The function $(g/f)' = \mathrm{Wr}(f,g)/f^2$ has zero residues and satisfies the assumptions of Lemma 9.1. This implies Lemma 9.2. □

Lemma 9.3. If $(\mu, t_1, \dots, t_m, z_1, \dots, z_{2m})$ is a solution of the Bethe ansatz equations and $\mu \notin \mathbb{Z}$, then there exist $s_1, \dots, s_m$ such that

$$\mathrm{Wr}\Big(\prod_{j=1}^{m} \theta(x-t_j,\tau), e^{-2\pi i\mu x}\prod_{j=1}^{m}\theta(x-s_j,\tau)\Big) = \mathrm{const}\, e^{-2\pi i\mu x}\prod_{a=1}^{2m}\theta(x-z_a,\tau) \quad (9.2)$$

Moreover, the function $\prod_{j=1}^{m}\theta(x-s_j,\tau)$ is unique up to multiplication by a constant.

Proof. The first multipliers of the functions $\prod_{j=1}^{m}\theta(x-t_j)$ and $e^{-2\pi i\mu x}\prod_{a=1}^{2m}\theta(x-z_a)$ are distinct. Hence there exists a unique theta-polynomial g satisfying the equation

$$\mathrm{Wr}\Big(\prod\nolimits_{j=1}^{m}\theta(x-t_j), g(x)\Big) = e^{-2\pi i\mu x}\prod\nolimits_{a=1}^{2m}\theta(x-z_a), \quad (9.3)$$

see Lemma 9.1 and Theorem 8.4. The theta-polynomial g has degree m and the first multiplier $A = e^{-2\pi i\mu}$, by Theorem 8.4. Such a function g has a presentation

$$g(x) = \mathrm{const}\, e^{-2\pi i\mu x}\prod\nolimits_{k=1}^{m}\theta(x-s_k) \quad (9.4)$$

for suitable $s_1, \dots, s_m \in \mathbb{C}$, by Lemma 7.3. □

Corollary 9.4. Under the assumptions of Lemma 9.3 we also have

$$\mathrm{Wr}\Big(e^{2\pi i\mu x}\prod_{j=1}^{m}\theta(x-t_j,\tau), \prod_{j=1}^{m}\theta(x-s_j,\tau)\Big) = \mathrm{const}\, e^{2\pi i\mu x}\prod_{a=1}^{2m}\theta(x-z_a,\tau). \quad (9.5)$$

In particular, if $s_1, \dots, s_m$ are distinct modulo the lattice $\mathbb{Z}+\tau\mathbb{Z}$ and any s_j is distinct from any z_a modulo the lattice $\mathbb{Z}+\tau\mathbb{Z}$, then $(-\mu, s_1, \dots, s_m, z_1, \dots, z_{2m})$ is a solution of the Bethe ansatz equations (6.2).

Proof. The corollary follows from formula (8.1). □

9.2. **Equivalence classes.**

Lemma 9.5. Let $(\mu, t_1, \dots, t_m, z_1, \dots, z_{2m})$ be a solution of the Bethe ansatz equations (6.2). Then

(i) for any $j \in \{1, \dots, m\}$ the points $(\mu \mathfrak{p} 2, t_1, \dots, t_j \pm \tau, \dots, t_m, z_1, \dots, z_{2m})$ and $(\mu, t_1, \dots, t_j \pm 1, \dots, t_m, z_1, \dots, z_{2m})$ are also solutions of the Bethe ansatz equations;

(ii) for any $k \in \{1, \dots, 2m\}$ the points $(\mu \pm 1, t_1, \dots, t_m, z_1, \dots, z_k \pm \tau, \dots, z_{2m})$ and $(\mu, t_1, \dots, t_m, z_1, \dots, z_k \pm 1, \dots, z_{2m})$ are also solutions of the Bethe ansatz equations.

□

Definition 9.6. We say that two solutions

$$(\mu, t_1, \dots, t_m, z_1, \dots, z_{2m}),\ (\mu', t_1', \dots, t_m', z_1', \dots, z_{2m}') \in \mathbb{C} \times \mathbb{C}^m/S_m \times \mathbb{C}^{2m}/S_{2m}$$

of the Bethe ansatz equations (6.2) are equivalent if one of them is obtained from the other by a sequence of transformations listed in Lemma 9.5.

Denote by Sol^1_m the set of equivalence classes of solutions $(\mu, t_1, \dots, t_m, z_1, \dots, z_{2m})$ of the Bethe ansatz equations (6.2) with $\mu \notin \mathbb{Z}$.

Theorem 9.7. We have a bijective correspondence

$$\beta \ : \ \mathrm{Sol}^1_m \ \to \ \mathrm{Pairs}^1_m$$

between the set Sol^1_m of equivalence classes of solutions of the Bethe ansatz equations with $\mu \notin \mathbb{Z}$ and the set Pairs^1_m of points of $(P_{m,1} \times (P_m - P_{m,1}))/\mathbb{Z}$ with generic first coordinates. The correspondence is given below in the proof of this lemma.

Proof. Take an equivalence class of solutions of the Bethe ansatz equations (6.2) with $\mu \notin \mathbb{Z}$ and choose a representative $(\mu, t_1, \dots, t_m, z_1, \dots, z_{2m})$. Then let $f(x) = \prod_{j=1}^m \theta(x - t_j, \tau)$ and let the function $g(x)$ be given by formula (9.4). Then the pair (f, g) determines a point of $(P_{m,1} \times (P_m - P_{m,1}))/\mathbb{Z}$ with generic first coordinate. Moreover, the equivalent solutions of the Bethe ansatz equations correspond under this construction to the pairs of theta-polynomials lying in the $\mathbb{Z}$-orbit of (f, g) in $P_{m,1} \times (P_m - P_{m,1})$.

Indeed, take, for example, the solution $(\mu - 2, t_1 + \tau, t_2, \dots, t_m, z_1, \dots, z_{2m})$. Then

$$\tilde f(x) = \theta(x - t_1 - \tau, \tau) \prod_{j=2}^m \theta(x - t_j, \tau) = e^{2\pi i x} \prod_{j=1}^m \theta(x - t_j, \tau)$$

and $\tilde g(x)$ is determined from the equation

$$\mathrm{Wr}(e^{2\pi i x} \prod_{j=1}^m \theta(x - t_j, \tau), \tilde g(x)) = e^{-2\pi i (\mu - 2) x} \prod_{a=1}^{2m} \theta(x - z_a, \tau).$$

Hence $\tilde g(x) = e^{2\pi i x} g(x)$ and the pairs (f, g) and $(\tilde f, \tilde g)$ lie in the same $\mathbb{Z}$-orbit.

Similarly take the solution $(\mu + 1, t_1, \dots, t_m, z_1 + \tau, z_2, \dots, z_{2m})$, then $\tilde f = f$ and $\tilde g$ is determined from the equation

$$\mathrm{Wr}(f, \tilde g) = e^{-2\pi i (\mu + 1) x} \theta(x - z_1 - \tau) \prod_{a=2}^{2m} \theta(x - z_a, \tau) = e^{-2\pi i \mu x} \prod_{a=1}^{2m} \theta(x - z_a, \tau).$$

Hence $\tilde g = g$ and $(f, g) = (\tilde f, \tilde g)$.

Conversely let $p \in (P_{m,1} \times P_m - P_{m,1})/\mathbb{Z}$ be a point with generic first coordinate. Choose a representative (f, g) and write

$$f(x) = \prod_{j=1}^m \theta(x - t_j, \tau), \qquad \mathrm{Wr}(f, g) = e^{-2\pi i \mu x} \prod_{a=1}^{2m} \theta(x - z_a, \tau), \tag{9.6}$$

for some $(\mu, t_1, \dots, t_m, z_1, \dots, z_{2m}, \tau)$, $\mu \notin \mathbb{Z}$. Then the function $\mathrm{Wr}(f, g)/f^2 = (g/f)'$ has zero residues and $(\mu, t_1, \dots, t_m, z_1, \dots, z_{2m}, \tau)$ is a solution of the Bethe ansatz equations (6.2) by Lemma 9.1. Different choices of the representative (f, g) of the point p or different choices of the presentations in (9.6) for the functions f and $\mathrm{Wr}(f, g)$ give equivalent solutions of the Bethe ansatz equations.

Indeed take, for example, another presentation

$$f(x) = \prod_{j=1}^m \theta(x - t_j, \tau) = \prod_{j=1}^m \theta(x - t'_j, \tau)$$

in (9.6). Then the solutions (μ, t, z) and (μ, t', z) are equivalent by Lemma 7.4. Or take another presentation

$$e^{-2\pi i \mu x} \prod_{a=1}^{2m} \theta(x - z_a, \tau) = e^{-2\pi i \mu' x} \prod_{a=1}^{2m} \theta(x - z'_a, \tau)$$

in (9.6). Then the solutions (μ, t, z) and (μ', t, z') are equivalent by Lemma 7.4. Finally, take another representative $(\tilde{f}, \tilde{g}) = (e^{2\pi ikx}f(x), e^{2\pi ikx}g(x))$ of the point p. Then

$$\tilde{f}(x) = \theta(x - t_1 - k\tau, \tau)\prod_{j=2}^{m}\theta(x - t_j, \tau), \qquad \mathrm{Wr}(\tilde{f}, \tilde{g}) = e^{-2\pi i(\mu-2k)x}\prod_{a=1}^{2m}\theta(x - z_a, \tau).$$

This gives a solution $(\mu - 2k, t_1 + k\tau, t_1, \dots, t_m, z_1, \dots, z_{2m})$, which is in the same equivalence class. □

By Theorem 9.7 we have a bijection $\beta : \mathrm{Sol}^1_m \to \mathrm{Pairs}^1_m$. We also have a subset $\mathrm{Pairs}_m \subset \mathrm{Pairs}^1_m$. Denote

$$\mathrm{Sol}_m = \beta^{-1}(\mathrm{Pairs}_m). \tag{9.7}$$

9.3. **Fundamental differential operators.** In formula (5.9) we introduced the fundamental differential operator $\mathcal{D}_{(\mu,t,z)}$ of a solution (μ, t, z) of the Bethe ansatz equations. The operator $\mathcal{D}_{(\mu,t,z)}$ is defined by the eigenvalues of the dynamical elliptic Bethe algebra on the eigenfunction $\Psi(\lambda_{12}, \mu, t, z, \tau)$, see (5.6). In Section 8.5 we introduced the fundamental differential operator $\mathcal{D}_{(f,g)}$ of a point $(f, g) \in (P_{m,1} \times (P_m - P_{m,1}))/\mathbb{Z}$ with generic first coordinate. In Theorem 9.7 we established a bijection $\beta : \mathrm{Sol}^1_m \to \mathrm{Pairs}^1_m$ between the set of equivalence classes of solutions (μ, t, z) of the Bethe ansatz equations with $\mu \notin \mathbb{Z}$ and the set of points of $(P_{m,1} \times (P_m - P_{m,1}))/\mathbb{Z}$ with generic first coordinates. Thus a solution (μ, t, z) with $\mu \notin \mathbb{Z}$ gets two fundamental differential operators: $\mathcal{D}_{(\mu,t,z)}$ and $\mathcal{D}_{(f,g)}$.

Theorem 9.8. We have $\mathcal{D}_{(\mu,t,z)} = \mathcal{D}_{(f,g)}$.

Proof. By Theorem 5.3,

$$\mathcal{D}_{(\mu,t,z)} = \big(\partial_x + (\ln u)'\big)\big(\partial_x - (\ln u)'\big)$$

where $u(x)$ is given by (5.10), while

$$\mathcal{D}_{(f,g)} = \big(\partial_x + (\ln u_1)'\big)\big(\partial_x - (\ln u_1)'\big),$$

where $u_1(x)$ is given by (8.5). We have $u = u_1$ by the correspondence described in Theorem 9.7. □

Corollary 9.9. The fundamental differential operators of equivalent solutions (μ, t, z), (μ', t', z') with $\mu, \mu' \notin \mathbb{Z}$ are equal.

9.4. **Analytic involution and Bethe ansatz.** Consider the analytic involution

$$\iota_{\mathrm{an}} : (P_{m,1} \times (P_m - P_{m,1}))/\mathbb{Z} \to (P_{m,1} \times (P_m - P_{m,1}))/\mathbb{Z}$$

defined in (7.12). By its construction in Section 7.5, the analytic involution restricts to an involution

$$\iota_{\mathrm{an}} \ : \ \mathrm{Pairs}_m \ \to \ \mathrm{Pairs}_m \tag{9.8}$$

on the set Pairs_m and induces an involution

$$\beta^{-1} \circ \iota_{\mathrm{an}} \circ \beta \ : \ \mathrm{Sol}_m \ \to \ \mathrm{Sol}_m \tag{9.9}$$

on the set Sol_m.

In other words, the involution in (9.9) is defined as follows. We start with a solution $(\mu, t_1, \dots, t_m, z_1, \dots, z_{2m})$, $\mu \notin \mathbb{Z}$, of the Bethe ansatz equations (6.2), construct the function $f = \prod_{j=1}^{m} \theta(x - t_j, \tau)$, determine the function $g = e^{-2\pi i \mu x} \prod_{j=1}^{m} \theta(x - s_j, \tau)$ from the equation $\mathrm{Wr}(f, g) = e^{-2\pi i \mu x} \prod_{a=1}^{2m} \theta(x - z_a, \tau)$. Then the involution in (9.9) sends the equivalence class of $(\mu, t_1, \dots, t_m, z_1, \dots, z_{rm})$ to the equivalence class of $(-\mu, s_1, \dots, s_m, z_1, \dots, z_{2m})$, see Lemma 9.3 and Corollary 9.4.

9.5. **Normal solutions.**

Definition 9.10. Given a fundamental parallelogram Λ of the lattice $\mathbb{Z} + \tau\mathbb{Z}$ acting on $\mathbb{C}$, a solution

$$(\mu, t_1, \dots, t_m, z_1, \dots, z_{2m}) \ \in \mathbb{C} \times \mathbb{C}^m / S_m \times \mathbb{C}^{2m} / S_{2m}$$

of the Bethe ansatz equations (6.2) will be called normal relative to Λ if $(t_1, \dots, t_m) \in \Lambda^m / S_m$ and $(z_1, \dots, z_{2m}) \in \Lambda^{2m} / S_{2m}$.

Lemma 9.11. Given a fundamental parallelogram Λ, then every equivalence class of solutions of the Bethe ansatz equations (6.2) has a unique normal solution. □

10. Bethe eigenfunctions for $(\mathbb{C}^2)^{\otimes n}[0]$

Notice that by Corollary 4.5 the dynamical elliptic Bethe algebra $\mathcal{B}^V(z_1, \dots, z_{2m}, \tau)$ does not change up to conjugation if the numbers $z_1, \dots, z_{2m}$ are shifted by elements of the lattice $\mathbb{Z} + \tau\mathbb{Z}$. We choose a fundamental parallelogram $\Lambda \subset \mathbb{C}$ of the lattiuce $\mathbb{Z} + \tau\mathbb{Z}$ and from now on *fix* $z_1, \dots, z_{2m}$ *to be distinct points of the open parallelogram* $\Lambda' \subset \Lambda$.

10.1. **Formula for eigenfunctions.** Let v_1, v_2 be the standard basis of the $\mathfrak{sl}_2$-module $\mathbb{C}^2$,

$$(e_{11} - e_{22})v_1 = v_1, \quad e_{21}v_1 = v_2 \quad e_{12}v_1 = 0,$$
$$(e_{11} - e_{22})v_2 = -v_2, \quad e_{21}v_2 = 0, \quad e_{12}v_2 = v_1.$$

A basis $(v_I)_I$ of $V = (\mathbb{C}^2)^{\otimes n}$ is labeled by subsets $I \subset \{1, \dots, 2m\}$, where

$$v_I = v_{j_1} \otimes \cdots \otimes v_{j_{2m}},$$

with $j_i = 2$ if $i \in I$ and $j_i = 1$ if $i \in \bar{I}$. The vectors $(v_I)_{|I|=m}$ form a basis of the zero weight subspace $V[0]$.

For $(\mu, t_1, \dots, t_m, z_1, \dots, z_{2m}) \in \mathbb{C}^{3m+1}$ define a $V[0]$-valued function $\Psi(\lambda_{12}, \mu, t, z, \tau)$ by the formula:

$$\Psi(\lambda_{12}, \mu, t, z, \tau) = e^{\pi i \mu \lambda_{12}} \sum\nolimits_{|I|=m} W_I(\lambda_{12}, t, z, \tau)\, v_I, \tag{10.1}$$

where for $I = \{i_1 < \cdots < i_m\} \subset \{1, \dots, 2m\}$ we define

$$W_I(\lambda_{12}, t, z) = \mathrm{Sym}_{t_1, \dots, t_m} \Big(\prod\nolimits_{j=1}^{m} \sigma(t_j - z_{i_j}, -\lambda_{12}, \tau) \Big) \tag{10.2}$$

and $\mathrm{Sym}_{t_1, \dots, t_m} (F(t_1, \dots, t_m)) = \sum_{\sigma \in S_m} F(t_{\sigma(1)}, \dots, t_{\sigma(m)})$.

We have the following periodicity property:

$$\Psi(\lambda_{12} + 1, \mu, t, z, \tau) \ = \ e^{\pi i \mu}\, \Psi(\lambda_{12}, \mu, t, z, \tau). \tag{10.3}$$

In the considered case of $V = (\mathbb{C}^2)^{\otimes n}$, Theorem 5.1 takes the following form.

Theorem 10.1 ([FV1]). Let t be a solution of the Bethe ansatz equations (6.2). Then the $V[0]$-valued function $\Psi(\lambda_{12},\mu,t,z,\tau)$ of λ_{12}, defined in (10.1), is such that

$$H_a(z,\tau)\,\Psi(\lambda_{12},\mu,t,z,\tau) \;=\; \frac{\partial\Phi}{\partial z_a}(\mu,t,z,\tau)\,\Psi(\lambda_{12},\mu,t,z,\tau), \qquad a=1,\dots,2m,$$

$$H_0(z,\tau)\Psi(\lambda_{12},\mu,t,z,\tau) \;=\; \frac{\partial\Phi}{\partial\tau}(\mu,t,z,\tau)\,\Psi(\lambda_{12},\mu,t,z,\tau).$$

10.2. **Eigenfunctions of equivalent solutions.** In Section 9.2 we introduced the notion of equivalent solutions of the Bethe ansatz equations (6.2)

Theorem 10.2. Let (μ,t,z,τ) and (μ',t',z,τ) be two equivalent solutions of the Bethe ansatz equations (6.2) with the same z, then $\Psi(\lambda_{12},\mu,t,z,\tau)=\Psi(\lambda_{12},\mu',t',z,\tau)$.

Proof. Let (μ,t,z,τ) be a solution. Consider an equivalent solution $(\mu-2,t_1,\dots,t_k+\tau,\dots,t_m,z,\tau)$, for some k, $1\leqslant k\leqslant m$. We show that the corresponding eigenfunctions are equal. Indeed the common factor $e^{\pi i\mu\lambda_{12}}$ in (10.1) is transformed into $e^{\pi i\mu\lambda_{12}}e^{-2\pi i\lambda_{12}}$, while the factor $\sigma(t_k-z_i,-\lambda_{12},\tau)$ in the product in (10.2) is transformed into $\sigma(t_k+\tau-z_i,-\lambda_{12},\tau)=e^{2\pi i\lambda_{12}}\sigma(t_k-z_i,-\lambda_{12},\tau)$. The two new factors are canceled and the corresponding eigenfunctions are equal. This proves the theorem. □

10.3. **Eigenfunctions and two involutions.** Let $\Psi(\lambda_{12})$ be a $V[0]$-valued eigenfunction of the dynamical elliptic Bethe algebra $\mathcal{B}^V(z_1,\dots,z_{2m})$ and with the fundamental differential operator $\mathcal{D}_\Psi$, see (5.7). Recall the Weyl group of $\mathfrak{sl}_2$, $W=\{\mathrm{id},s\}$. We have

$$\mathcal{D}_{s(\Psi)}=\mathcal{D}_\Psi,$$

since $\mathcal{B}^V(z_1,\dots,z_{2m})$ is Weyl group invariant by Lemma 4.11. On the other hand, let $(\mu,t,z)\in\mathrm{Sol}_m$ be an equivalence class of solutions of the Bethe ansatz equation (6.2), see (9.7). Let $\Psi(\lambda_{12},\mu,t,z)$ be the associated eigenfunction of the dynamical elliptic Bethe algebra $\mathcal{B}^V(z_1,\dots,z_{2m})$ with the fundamental differential operator $\mathcal{D}_{(\mu,t,z)}$, see (5.9). Both $\Psi(\lambda_{12},\mu,t,z)$ and $\mathcal{D}_{(\mu,t,z)}$ are well-defined, see Theorem 10.2.

Recall the analytic involution

$$\beta^{-1}\circ\iota_{\mathrm{an}}\circ\beta\;:\;\mathrm{Sol}_m\;\to\;\mathrm{Sol}_m,$$

defined in (9.9). Let $(-\mu,s,z)$ be the image of (μ,t,z) under this involution, see Section 9.4. Let $\Psi(\lambda_{12},-\mu,s,z)$ be the associated eigenfunction. Then

$$\mathcal{D}_{(-\mu,s,z)}=\mathcal{D}_{(\mu,t,z)},$$

by Theorem 8.14. Now the following three eigenfunctions have the same fundamental differential operator: $\Psi(\lambda_{12},\mu,t,z)$, $s(\Psi(\lambda_{12},\mu,t,z))$, $\Psi(\lambda_{12},-\mu,s,z)$.

Theorem 10.3. We have

$$s(\Psi(\lambda_{12},\mu,t,z))=\mathrm{const}\;\;\Psi(\lambda_{12},-\mu,s,z).$$

In other words, the Weyl involution coincides with the analytic involution.

The theorem is proved in Section 12.6.

Theorem 10.4. Assume that $(\mu, t, z), (\mu', t', z) \in \mathrm{Sol}_m$ and the two Bethe eigenfunctions $\Psi(\lambda_{12}, \mu, t, z)$ and $\Psi(\lambda_{12}, \mu', t', z)$ have the same eigenvalues for every element of the dynamical elliptic Bethe algebra $\mathcal{B}^V(z_1, \dots, z_{2m})$. Then either $(\mu, t, z) = (\mu', t', z)$ or (μ, t, z) is the image of (μ', t', z) under the analytic involution $\beta^{-1} \circ \iota_{\mathrm{an}} \circ \beta$.

Theorem 10.4 says that the dynamical elliptic Bethe algebra separates the Weyl group orbits of the Bethe eigenfunctions.

Proof. The assumptions of the theorem mean that $\Psi(\lambda_{12}, \mu, t, z)$ and $\Psi(\lambda_{12}, \mu', t', z)$ have the same fundamental differential operators. Now the theorem follows from Theorem 8.14. □

11. Elliptic Wronski map

11.1. **Wronski map.** Define the *elliptic Wronski map*

$$P_m \times P_m \to P_{2m}, \qquad (f, g) \mapsto \mathrm{Wr}(f(x), g(x)). \tag{11.1}$$

The group $\mathbb{C}$ acts on $P_m \times P_m$ by the operators $L_\nu \times L_\nu$, $\nu \in \mathbb{C}$. The maps $L_{2\nu}$, $\nu \in \mathbb{C}$, define an action of $\mathbb{C}$ on P_{2m}. The Wronski map commutes with the actions of $\mathbb{C}$ on $P_m \times P_m$ and P_{2m}:

$$\mathrm{Wr} \circ (L_\nu \times L_\nu) = L_{2\nu} \circ \mathrm{Wr}, \tag{11.2}$$

see formula (8.1),

We are interested in the elliptic Wronski map

$$\mathrm{Wr} : P_{m,1} \times (P_m - P_{m,1}) \ \to \ P_{2m} - P_{2m,1}. \tag{11.3}$$

This is a holomorphic map between $2m+1$-dimensional complex manifolds.

Notice that the map in (11.3) induces a map

$$(P_{m,1} \times (P_m - P_{m,1}))/\mathbb{Z} \ \to \ (P_{2m} - P_{2m,1})/\mathbb{Z}, \tag{11.4}$$

see (11.2).

11.2. **Wronski map has nonzero Jacobian.**

Lemma 11.1. The Jacobian of the Wronski map $\mathrm{Wr} : P_{m,1} \times (P_m - P_{m,1}) \to P_{2m} - P_{2m,1}$ is nonzero at generic points of $P_{m,1} \times (P_m - P_{m,1})$.

Proof. Let (f, g) be a generic point, $f(x) = \prod_{j=1}^m \theta(x - t_j, \tau)$, $g(x) = e^{-2\pi i \mu x} \prod_{j=1}^m \theta(x - s_j, \tau)$ with some distinct $t_1, \dots, t_m$, $s_1, \dots, s_m$ modulo the lattice $\mathbb{Z} + \tau\mathbb{Z}$. The parameters μ, $t_1, \dots, t_m$, $s_1, \dots, s_m$ are local coordinates on $P_{m,1} \times (P_m - P_{m,1})$.

Let $e^{-2\pi i \mu x} \prod_{a=1}^{2m} \theta(x - z_a, \tau) \in P_{2m} - P_{2m,1}$. The parameters μ, $z_1, \dots, z_{2m}$ are local coordinates on $P_{2m} - P_{2m,1}$.

In order to prove the lemma it is enough to prove that for fixed generic μ, the map

$$\Big(\prod_{j=1}^m \theta(x - t_j, \tau), e^{-2\pi i \mu x} \prod_{j=1}^m \theta(x - s_j, \tau)\Big) \tag{11.5}$$
$$\mapsto \quad \mathrm{Wr}(f, g) = e^{-2\pi i \mu x} \prod_{a=1}^{2m} \theta(x - z_a, \tau),$$

sending $(t_1, \dots, t_m, s_1, \dots, s_m)$ to $(z_1, \dots, z_{2m})$ has nonzero Jacobian at a generic point $(t_1, \dots, t_m, s_1, \dots, s_m)$.

First consider the multiplication map

$$\big(\prod_{j=1}^m \theta(x-t_j,\tau), e^{-2\pi i\mu x}\prod_{j=1}^m \theta(x-s_j,\tau)\big) \\ \mapsto e^{-2\pi i\mu x}\prod_{j=1}^m \theta(x-t_j,\tau)\prod_{j=1}^m \theta(x-s_j,\tau) \qquad (11.6)$$

sending $(t_1,\dots,t_m,s_1,\dots,s_m)$ to $(z_1,\dots,z_{2m})=(t_1,\dots,t_m,s_1,\dots,s_m)$ which is a local isomorphism. For large μ the map in (11.5) is a small deformation of the map in (11.6). Indeed

$$\mathrm{Wr}\,\big(\prod_{j=1}^m \theta(x-t_j,\tau), e^{-2\pi i\mu x}\prod_{j=1}^m \theta(x-s_j,\tau)\big)$$
$$= -2\pi i\mu\, e^{-2\pi i\mu x}\Big(\prod_{j=1}^m \theta(x-t_j)\prod_{j=1}^m \theta(x-s_j) - \frac{1}{2\pi i\mu}\,\mathrm{Wr}(\prod_{k=1}^j \theta(x-t_j),\prod_{j=1}^m \theta(x-s_j))\Big).$$

Hence for large μ the Jacobian of the map in (11.5) is not identically equal to zero. □

Another proof of Lemma 11.1 can extracted from the proof of Theorem 11.2 below.

11.3. **Labels.** Choose $w\in\mathbb{C}$. Let $\Lambda=\Lambda_w\subset\mathbb{C}$ be the fundamental parallelogram with vertices $w, w{+}1, w{+}\tau, w{+}1{+}\tau$ and with boundary intervals $[w,w{+}1), [w,w{+}\tau)$ included and boundary intervals $[w{+}1,w{+}1{+}\tau], [w{+}\tau,w{+}1{+}\tau]$ excluded. Denote by $\bar\Lambda$ the closure of Λ, by $\partial\bar\Lambda$ the boundary of $\bar\Lambda$ and by Λ' the open parallelogram $\bar\Lambda-\partial\bar\Lambda$. The parallelogram Λ is a fundamental domain for the $\mathbb{Z}+\tau\mathbb{Z}$-action on $\mathbb{C}$.

For any theta-polynomial $h\in P_k$ (considered up to multiplication by a nonzero constant) there exist unique $u=(u_1,\dots,u_k)\in\Lambda^k/S_k$ and $\mu\in\mathbb{C}$ such that

$$h(x)=e^{2\pi i\mu x}\prod_{j=1}^k\theta(x-u_j,\tau).$$

The pair $(u,\mu)\in\Lambda^k/S_k\times\mathbb{C}$ will be called the *coordinates of* $h(x)$ relative to the fundamental parallelogram Λ. The number μ will be called the *label* of h relative to Λ and denoted $l(h)$.

11.4. **Labeled preimage.** Recall the Wronski map

$$\mathrm{Wr}: P_{m,1}\times(P_m-P_{m,1}) \;\to\; P_{2m}-P_{2m,1}.$$

For $h\in P_{2m}-P_{2m,1}$ and $k\in\mathbb{Z}$ define the *labeled preimage* of h with label k as the set

$$\mathrm{Wr}_k^{-1}(h) := \{(f,g)\in P_{m,1}\times(P_m-P_{m,1}) \mid \mathrm{Wr}(f,g)=h,\ l(f)=k\}. \qquad (11.7)$$

We have

$$\mathrm{Wr}^{-1}(h)=\cup_{k\in\mathbb{Z}}\mathrm{Wr}_k^{-1}(h),$$

and $\mathrm{Wr}_k^{-1}(h)\cap\mathrm{Wr}_{k'}^{-1}(h)=\emptyset$ if $k\neq k'$.

Theorem 11.2. Choose a fundamental parallelogram Λ and $z=(z_1,\dots,z_{2m})\in(\Lambda')^{2m}/S_{2m}$ with distinct coordinates. Then there exists $N>0$, such that for any $h\in P_{2m}-P_{2m,1}$ of the form

$$h(x)=e^{-2\pi i\mu x}\prod_{a=1}^{2m}\theta(x-z_a,\tau), \qquad (11.8)$$

with $|\operatorname{Im}\mu| > N$, and any $k \in \mathbb{Z}$, the set $\mathrm{Wr}_k^{-1}(h)$ consists of exactly $\binom{2m}{m}$ points. Moreover, in that case any $(f,g) \in \mathrm{Wr}_k^{-1}(h)$ has the form

$$f(x) = e^{2\pi i k x}\prod_{j=1}^{m}\theta(x-t_j,\tau), \qquad g(x) = e^{-2\pi i(\mu+k)x}\prod_{j=1}^{m}\theta(x-s_j,\tau), \quad (11.9)$$

where $t = (t_1,\dots,t_m) \in (\Lambda')^m/S_m$, $s = (s_1,\dots,s_m) \in (\Lambda')^m/S_m$, and all the numbers $t_1,\dots,t_m$, $s_1,\dots,s_m$, $z_1,\dots,z_{2m}$ are pairwise distinct.

Theorem 11.2 is proved in Section 11.5.

Corollary 11.3. Under the assumptions of Theorem 11.2, the points $(f,g) \in \mathrm{Wr}_k^{-1}(h)$ have generic first and second coordinates, and any two points of $\mathrm{Wr}^{-1}(h)$ lie in different $\mathbb{Z}$-orbits. Moreover, at each point $(f,g) \in \mathrm{Wr}^{-1}(h)$ the Jacobian of the Wronski map is nonzero. □

Remark. Some additional asymptotic structure of the points of $\mathrm{Wr}_k^{-1}(h)$ as $\operatorname{Im}\mu \to \infty$ can be observed in the proof of Theorem 11.2.

11.5. **Proof of Theorem 11.2.** For $\mu \in \mathbb{C} - \mathbb{Z}$ consider the multiplication map

$$M : P_{m,1} \times P_{m,e^{-2\pi i\mu}} \to P_{2m,e^{-2\pi i\mu}}, \quad (f,g) \mapsto fg,$$

and its fiber

$$M^{-1}(h) = \{(f,g) \in P_{m,1} \times P_{m,e^{-2\pi i\mu}} \mid fg = h\}$$

over a point $h = e^{-2\pi i\mu x}\prod_{j=1}^{2m}\theta(x-z_j)$ with $(z,\mu) \in (\Lambda')^{2m}/S_{2m} \times (\mathbb{C}-\mathbb{Z})$ and z with distinct coordinates. Then the points of $M^{-1}(h)$ are labeled by the pairs (I,k), where $I \subset \{1,\dots,2m\}$ is an m-element subset and $k \in \mathbb{Z}$. The corresponding point $(f_{I,k}, g_{I,k}) \in M^{-1}(h)$ has the form

$$f_{I,k} = e^{2\pi i k x}\prod_{j\in I}\theta(x-z_j,\tau), \qquad g_{I,k} = e^{-2\pi i(\mu+k)x}\prod_{j\in \bar I}\theta(x-z_j,\tau),$$

where $\bar I$ is the the complement to I in $\{1,\dots,2m\}$.

Recall $\bar\Lambda \subset \mathbb{C}$, the closure of the parallelogram Λ. For $k \in \mathbb{Z}$ denote

$$V_k = \{f \in P_{m,1} \mid f = e^{2\pi i k x}\prod_{j=1}^{m}\theta(x-t_j,\tau) \text{ where } t = (t_1,\dots,t_m) \in \bar\Lambda^m/S_m\},$$

$$V_{\mu,k} = \{g \in P_{m,e^{-2\pi i\mu}} | \, g = e^{-2\pi i(\mu+k)x}\prod_{j=1}^{m}\theta(x-s_j,\tau) \text{ where } s = (s_1,\dots,s_m) \in \bar\Lambda^m/S_m\}.$$

We see that for any $k_1,k_2 \in \mathbb{Z}$ the sets $V_{k_1}, V_{\mu,k_2}, V_{k_1} \times V_{\mu,k_2}$ are compact subsets of $P_{m,1}$, $P_{m,e^{-2\pi i\mu}}$, $P_{m,1} \times P_{m,e^{-2\pi i\mu}}$, respectively, and

$$P_{m,1} = \cup_{k_1\in\mathbb{Z}} V_{k_1}, \qquad P_{m,e^{-2\pi i\mu}} = \cup_{k_2\in\mathbb{Z}} V_{\mu,k_2} \qquad P_{m,1} \times P_{m,e^{-2\pi i\mu}} = \cup_{k_1,k_2\in\mathbb{Z}} V_{k_1} \times V_{\mu,k_2}.$$

The intersection $M^{-1}(h) \cap (V_{k_1} \times V_{\mu,k_2})$ consists of $\binom{2m}{m}$ points if $k_1 = k_2$ and is empty if $k_1 \neq k_2$.

Consider the Wronski map

$$\mathrm{Wr} : P_{m,1} \times P_{m,e^{-2\pi i\mu}} \to P_{2m,e^{-2\pi i\mu}}, \quad (f,g) \mapsto \mathrm{Wr}(f,g),$$

and the fiber $\mathrm{Wr}^{-1}(h) = \{(f,g) \in P_{m,1} \times P_{m,e^{-2\pi i\mu}} | \; \mathrm{Wr}(f,g) = h\}$ over the same point $h = e^{-2\pi i\mu x}\prod_{j=1}^{2m}\theta(x-z_j)$.

Let $(f,g) \in V_{k_1} \times V_{\mu,k_2}$ with $f = e^{2\pi i k_1 x}\prod_{j=1}^m \theta(x-t_j)$ for some $(t_1,\dots,t_j) \in \bar\Lambda^m/S_m$ and $g = e^{-2\pi i(\mu+k_2)x}\prod_{j=1}^m \theta(x-s_j)$ for some $(s_1,\dots,s_m) \in \bar\Lambda^m/S_m$. Then

$$\mathrm{Wr}(f,g) = -2\pi i(\mu+k_2+k_1)\, e^{-2\pi i(\mu+k_2-k_1)x} \times \Big(\prod_{j=1}^m \theta(x-t_j)\prod_{j=1}^m \theta(x-s_j) - \frac{1}{2\pi i(\mu+k_1+k_2)}\,\mathrm{Wr}\,\big(\prod_{j=1}^m \theta(x-t_j), \prod_{j=1}^m \theta(x-s_j)\big)\Big). \tag{11.10}$$

Since the functions $f, g, \mathrm{Wr}(f,g)$ are considered up to multiplication by nonzero numbers, we may ignore the first factor $-2\pi i(\mu+k_2+k_1)$ in the right-hand side.

Let us analyze the last factor in (11.10). Let

$$G(x,t,s,v) \;=\; \prod_{j=1}^m \theta(x-t_j)\prod_{j=1}^m \theta(x-s_j) - v\,\mathrm{Wr}\,\big(\prod_{j=1}^m \theta(x-t_j), \prod_{j=1}^m \theta(x-s_j)\big). \tag{11.11}$$

The function $G(x,t,s,v)$, as a function of x, has the same multipliers as the function $\prod_{j=1}^m \theta(x-t_j)\prod_{j=1}^m \theta(x-s_j)$, see the proof of Lemma 8.2. Namely, the first multiplier of $G(x,t,s,v)$ is 1 and the second is $e^{2\pi i\sum_{j=1}^m (t_j+s_j)}$. By Lemma 7.3, for any (t,s,v) we have

$$G(x,t,s,v) = c(t,s,v)\prod_{j=1}^{2m} \theta(x-u_j(t,s,v)) \tag{11.12}$$

for some $c(t,s,v) \in \mathbb{C}$, $u(t,s,v) = (u_1(t,s,v),\, \dots,\, u_{2m}(t,s,v)) \in \mathbb{C}^{2m}/S_{2m}$ and

$$e^{2\pi i\sum_{j=1}^{2m} u_j(t,s,v)} = e^{2\pi i\sum_{j=1}^m (t_j+s_j)}. \tag{11.13}$$

The pair $(c(t,s,v), u(t,s,v))$ is not unique, see Lemma 7.4.

Lemma 11.4. Consider $(\Lambda')^{2m}$ with coordinates $t_1,\dots,t_m,s_1,\dots,s_m$. Let C be a compact subset of $(\Lambda')^{2m}$ which is disjoint from all diagonals and invariant with respect to the $S_m \times S_m$-action. Then there exists $\delta > 0$, such that the pair $(c(t,s,v), u(t,s,v))$ in (11.12) can be chosen so that

$$\mathbb{C}^m/S_m \times \mathbb{C}^m/S_m \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}^{2m}/S_{2m}, \quad (t,s,v) \to (c(t,s,v), u(t,s,v)), \tag{11.14}$$

is a well-defined holomorphic map for $(t,s) \in C/(S_m \times S_m)$ and $v \in \mathbb{C}$, $|v| < \delta$, and

(i) $u(t,s,v) = (u_1(t,s,v),\, \dots,\, u_{2m}(t,s,v))$ has distinct coordinates;
(ii) $(c(t,s,0), u(t,s,0)) = (1,(t,s))$;
(iii) for any v with $|v| < \delta$ the restriction map

$$C/(S_m \times S_m) \times \{v\} \to \mathbb{C}^{2m}/S_{2m}, \qquad (t,s,v) \to u(t,s,v), \tag{11.15}$$

has nonzero Jacobian.

Proof of Lemma 11.4. We first construct a certain holomorphic map

$$C \times \{v \in \mathbb{C} \mid |v| \ll 1\} \to \mathbb{C} \times \mathbb{C}^{2m}, \qquad ((t,s),v) \mapsto (\tilde c(t,s,v), \tilde u(t,s,v)), \tag{11.16}$$

as follows.

For $(t, s) \in C$, the numbers $x = t_j$, $j = 1, \dots, m$, and $x = s_j$, $j = 1, \dots, m$, are simple zeros of the function $G(x, t, s, 0)$. By the implicit function theorem there exist unique holomorphic functions $\tilde{u}_j(t, s, v)$, $j = 1, \dots, m$, defined in a neighborhood of $(t, s, 0)$ and such that $G(\tilde{u}_j(t, s, v), t, s, v) = 0$, $\tilde{u}_j(t, s, 0) = t_j$. Similarly, there exist unique holomorphic functions $\tilde{u}_j(t, s, v)$, $j = m+1, \dots, 2m$, defined in a neighborhood of $(t, s, 0)$ and such that $G(\tilde{u}_j(t, s, v), t, s, v) = 0$, $\tilde{u}_j(t, s, 0) = s_{j-m}$.

Since C is compact, there is a $\delta > 0$ such that $\tilde{u}(t, s, v) = (\tilde{u}_1(t, s, v), \dots, \tilde{u}_{2m}(t, s, v))$ is holomorphic for $(t, s) \in C$ and $v \in \mathbb{C}$, $|v| < \delta$, and has distinct coordinates. Denote $G(x, t, s, v) = \prod_{j=1}^{2m} \theta(x - \tilde{u}_j(t, s, v))$ and define $\tilde{c}(x, t, s, v)$ by the formula

$$G(x, t, s, v) = \tilde{c}(x, t, s, v) H(x, t, s, v). \tag{11.17}$$

Since $G(x, t, s, v)$ and $H(x, t, s, v)$ have the same zeros, the function $\tilde{c}(x, t, s, v)$ is holomorphic in x, t, s, v and has the from

$$\tilde{c}(x, t, s, v) = \hat{c}(t, s, v) e^{2\pi i k(t,s,v)x} \tag{11.18}$$

for some $\hat{c}(t, s, v) \in \mathbb{C}$, $k(t, s, v) \in \mathbb{Z}$, such that $k(t, s, 0) = 0$ and $\hat{c}(t, s, 0) = 1$. We know that the second multiplier of the function $G(x, t, s, v)$, as a function of x, is $e^{2\pi i \sum_{j=1}^{m}(t_j + s_j)}$. From (11.18) we conclude that the second multiplier of $G(x, t, s, v)$ is $e^{2\pi i k(t,s,v)\tau + 2\pi i \sum_{j=1}^{2m} u_j(t,s,v)}$. Since $\sum_{j=1}^{2m} u_j(t, s, v)$ is continuous, we conclude that $k(t, s, u) = 0$ and $\tilde{c}(x, t, s, v)$ in (11.17) does not depend on x. The map in (11.16) is constructed. Clearly, we can choose the positive δ so small that for any v with $|v| < \delta$, the restriction map

$$C \times \{v\} \to \mathbb{C}^{2m}, \qquad ((t, s), v) \mapsto \tilde{u}(t, s, v),$$

has nonzero Jacobian, since it is a deformation of the identity map.

For any permutations $\sigma, \eta \in S_m$ and $j = 1, \dots, m$, we clearly have

$$\begin{aligned} &\tilde{u}_j(t_{\sigma_1}, \dots, t_{\sigma_m}, s_{\eta_1}, \dots, s_{\eta_m}, v) = \tilde{u}_{\sigma_j}(t_1, \dots, t_m, s_1, \dots, s_m, v), \\ &\tilde{u}_{m+j}(t_{\sigma_1}, \dots, t_{\sigma_m}, s_{\eta_1}, \dots, s_{\eta_m}, v) = \tilde{u}_{m+\eta_j}(t_1, \dots, t_m, s_1, \dots, s_m, v). \\ &\tilde{c}(t_{\sigma_1}, \dots, t_{\sigma_m}, s_{\eta_1}, \dots, s_{\eta_m}, v) = \tilde{c}(t_1, \dots, t_m, s_1, \dots, s_m, v). \end{aligned}$$

Hence the map in (11.16) projects to the required map in (11.14) after factorizing the preimage of the map in (11.16) by $S_m \times S_m$. The lemma is proved. □

Recall our $z = (z_1, \dots, z_{2m}) \in (\Lambda')^{2m}/S_{2m}$ with distinct coordinates. Let $\epsilon_1 > 0$ be the distance from the set $\{z_1, \dots, z_{2m}\} \subset \Lambda'$ to the boundary $\partial\bar{\Lambda}$ of $\bar{\Lambda}$. Let $\epsilon_2 = \min_{a,b,\, 1 \leqslant a < b \leqslant 2m} |z_a - z_b|$. Choose any ϵ with $0 < \epsilon < \min\{\epsilon_1/2, \epsilon_2/2\}$.

Denote by Z the set of all points $(t, s) = (t_1, \dots, t_m, s_1, \dots, s_m) \in \bar{\Lambda}^{2m}$, which have at least two equal coordinates or at least one coordinate lying in $\partial\bar{\Lambda}$. Clearly, Z is closed and invariant with respect to the $S_m \times S_m$-action on $\bar{\Lambda}^{2m}$.

Lemma 11.5. There exists a neighborhood U of Z in $\bar{\Lambda}^{2m}$ and a number $\delta' > 0$ with the following properties:

(i) U is $S_m \times S_m$-invariant;
(ii) the point z does not lie in the closure $\overline{U/S_{2m}}$ of U/S_{2m};
(iii) for any $(t, s) \in U$, $v \in \mathbb{C}$ with $|v| < \delta'$, the function $F(x, t, s, v)$ has a multiple zero, or has a zero with the distance less than ϵ to the boundary $\partial\bar{\Lambda}$, or has two distinct zeros with the distance between them less than ϵ.

Proof. The lemma clearly follows from the fact that $\bar\Lambda^{2m}$ is compact and $F(x,t,s,v)$ is holomorphic. ☐

Let U, δ' be as in Lemma 11.5. Define the compact set $C = \bar\Lambda^{2m} - U$. The set C satisfies the assumptions of Lemma 11.4. By Lemma 11.4 there exists δ, $0 < \delta < \delta'$, such that the function $u(t,s,v) = (u_1(t,s,v), \ldots, u_{2m}(t,s,v))$ of Lemma 11.4 has the following properties:

(iv) For any v with $|v| < \delta$, the equation $u(t,s,v) = z$ has exactly $\binom{2m}{m}$ distinct solutions $(t(v), s(v)) \in C/(S_m \times S_m$. Each of the solutions is a holomorphic function of v. The solutions are labeled by m-element subsets $I \subset \{1,\ldots,2m\}$. The corresponding solution $(t_I(v), s_I(v))$ is such that

$$t_I(0) = (z_j)_{j\in I}, \qquad s_I(0) = (z_j)_{j\in \bar I}. \tag{11.19}$$

(v) For $(t,s) \in C/(S_m \times S_m)$ and v with $|v| < \delta$, each coordinate of $u(t,s,v)$ lies in the ϵ-neighborhood of the parallelogram $\bar\Lambda$.

Let us return to the proof of Theorem 11.2. Let $h = e^{-2\pi i\mu x}\prod_{a=1}^{2m}\theta(x-z_a)$ as before. Assume that $\mu \in \mathbb{C}$ is such that $|\operatorname{Im}\mu| > \delta^{-1}$, then $|2\pi i(\mu+k_2+k_1)|^{-1} < \delta$ for any $k_1, k_2 \in \mathbb{Z}$. Then formula (11.10), property (iii) of Lemma 11.5 and properties (iv-v) above show that the intersection $\mathrm{Wr}^{-1}(h) \cap (V_{k_1} \times V_{\mu,k_2})$ is empty if $k_1 \neq k_2$ and consists of exactly $\binom{2m}{m}$ points if $k_1 = k_2$, moreover, those points lie in the ϵ-neighborhood of points $((z_j)_{j\in I}, (z_j)_{j\in\bar I})$.

This proves all of the statements of Theorem 11.2 except the last statement that all of the numbers $t_1,\ldots,t_m,\ s_1,\ldots,s_m,\ z_1,\ldots,z_{2m}$ are distinct, see the theorem. We already know that

(1) $\mathrm{Wr}(e^{2\pi ikx}\prod_{j=1}^m\theta(x-t_j), e^{-2\pi i(\mu+k)x}\prod_{j=1}^m\theta(x-s_j)) = e^{-2\pi i\mu x}\prod_{j=1}^{2m}\theta(x-z_j)$,
(2) all the numbers $t_1,\ldots,t_m,\ s_1,\ldots,s_m$ are distinct,
(3) all the numbers $z_1,\ldots,z_{2m}$ are distinct.

This implies that $t_1,\ldots,t_m,\ s_1,\ldots,s_m,\ z_1,\ldots,z_{2m}$ are distinct. Theorem 11.2 is proved.

12. Applications of Theorem 11.2

12.1. **Counting ratios of theta-polynomials.** Fix a fundamental parallelogram $\Lambda \subset \mathbb{C}$. Consider the ratio F of two theta-polynomials of degree m with m simple poles in Λ. Then the function F can be written uniquely in the form

$$F = g/f, \qquad f = \prod_{j=1}^m \theta(x-t_j,\tau), \tag{12.1}$$

where $t = (t_1,\ldots,t_m)$ is a point of Λ^m/S_m with distinct coordinates and g is a theta-polynomial of degree m. The derivative $F' = \mathrm{Wr}(f,g)/f^2$ can be written uniquely in the form

$$F' = e^{-2\pi i\mu x}\frac{\prod_{a=1}^{2m}\theta(x-z_a,\tau)}{\prod_{j=1}^m\theta(x-t_j,\tau)^2} \tag{12.2}$$

for some $\mu \in \mathbb{C}$ and $z = (z_1,\ldots,z_{2m}) \in \Lambda^{2m}/S_{2m}$. By assumptions the numerator and denominator of this ratio have no common zeros.

Theorem 12.1. Let $z = (z_1, \dots, z_{2m}) \in (\Lambda')^{2m}/S_{2m}$ have distinct coordinates. Then there exists $N > 0$, such that for any $\mu \in \mathbb{C}$ with $|\,\mathrm{Im}\,\mu| > N$ there exist exactly $\binom{2m}{m}$ functions $F(x)$ as in (12.1) with the derivative as in (12.2), up to proportionality.

Proof. By construction, these functions $F = g/f$ are in the bijective correspondence with the points $(f, g) \in \mathrm{Wr}_0^{-1}(h)$, where $h = e^{-2\pi i\mu x}\prod_{a=1}^{2m}\theta(x - z_a)$. Now Theorem 12.1 is a corollary of Theorem 11.2, see also Lemma 11.4. □

12.2. **Counting normal solutions of Bethe ansatz equations.** Choose $z = (z_1, \dots, z_{2m}) \in (\Lambda')^{2m}/S_{2m}$ with distinct coordinates. By Theorem 11.2 there exists $N > 0$, such that for any $h \in P_{2m} - P_{2m,1}$ of the form

$$h(x) = e^{-2\pi i\mu x}\prod_{a=1}^{2m}\theta(x - z_a, \tau), \tag{12.3}$$

with $|\,\mathrm{Im}\,\mu| > N$, the set $\mathrm{Wr}_0^{-1}(h)$ consists of exactly $\binom{2m}{m}$ points. In that case any $(f, g) \in \mathrm{Wr}_0^{-1}(h)$ has the form

$$f(x) = \prod_{j=1}^{m}\theta(x - t_j, \tau), \qquad g(x) = e^{-2\pi i\mu x}\prod_{j=1}^{m}\theta(x - s_j, \tau), \tag{12.4}$$

where $t = (t_1, \dots, t_m) \in (\Lambda')^m/S_m$, $s = (s_1, \dots, s_m) \in (\Lambda')^m/S_m$, and all the numbers $t_1, \dots, t_m$, $s_1, \dots, s_m$, $z_1, \dots, z_{2m}$ are pairwise distinct.

In particular, this means that

$$\mathrm{Wr}\Big(\prod_{j=1}^{m}\theta(x - t_j, \tau), e^{-2\pi i\mu x}\prod_{j=1}^{m}\theta(x - s_j, \tau)\Big) = e^{-2\pi i\mu x}\prod_{a=1}^{2m}\theta(x - z_a, \tau)\Big) \tag{12.5}$$

and

$$\mathrm{Wr}\Big(e^{2\pi i\mu x}\prod_{j=1}^{m}\theta(x - t_j, \tau), \prod_{j=1}^{m}\theta(x - s_j, \tau)\Big) = e^{2\pi i\mu x}\prod_{a=1}^{2m}\theta(x - z_a, \tau)\Big). \tag{12.6}$$

By Lemmas 9.1 - 9.3 equation (12.5) implies that $t = (t_1, \dots, t_m)$ is a solution of the Bethe ansatz equations

$$2\pi i\mu + 2\sum_{\ell,\,\ell\neq j}\rho(t_j - t_\ell, \tau) - \sum_{a=1}^{2m}\rho(t_j - z_a, \tau) = 0, \quad j = 1, \dots, m, \tag{12.7}$$

and equation (12.6) implies that $s = (s_1, \dots, s_m)$ is a solution of the Bethe ansatz equations

$$-2\pi i\mu + 2\sum_{\ell,\,\ell\neq j}\rho(s_j - s_\ell, \tau) - \sum_{a=1}^{2m}\rho(s_j - z_a, \tau) = 0, \quad j = 1, \dots, m. \tag{12.8}$$

Hence the equivalence classe of the solution $(\mu, t_1, \dots, t_m, z_1, \dots, z_{2m})$ and the equivalence class of the solution $(-\mu, s_1, \dots, s_m, z_1, \dots, z_{2m})$ belong to the set Sol_m and one of them is the image of the other under the analytic involution $\beta^{-1}\circ\iota_{\mathrm{an}}\circ\beta$, see (9.9).

Given $\mu \in \mathbb{C}$ and $z = (z_1, \dots, z_{2m}) \in \Lambda^{2m}/S_{2m}$ denote by $B(\mu, z)$ the set of equivalence classes of solutions $(\mu', t_1', \dots, t_m', z_1, \dots, z_{2m})$ of the Bethe ansatz equations (6.2), which have representatives of the form $(\mu, t_1, \dots, t_m, z_1, \dots, z_{2m})$ with $(t_1, \dots, t_m) \in \Lambda^m/S_m$.

Theorem 12.2. Let $z = (z_1, \dots, z_{2m}) \in (\Lambda')^{2m}/S_{2m}$ have distinct coordinates. Then there exists $N > 0$, such that for any μ with $|\mu| > N$, each of the sets $B(\mu, z)$ and $B(-\mu, z)$ consist of exactly $\binom{2m}{m}$ points. Moreover, there is a bijection $B(z, \mu) \to B(z, -\mu)$, given by the analytic involution $\beta^{-1} \circ \iota_{\mathrm{an}} \circ \beta$, which combines the points of these two sets into $\binom{2m}{m}$ pairs $(t, s) \in B(\mu, z) \times B(-\mu, z)$ so that the pairs $(\prod_{j=1}^{m} \theta(x - t_j), e^{-2\pi i \mu x} \prod_{j=1}^{m} \theta(x - s_j))$ list all the points of the set $\mathrm{Wr}_0^{-1}(h)$, see (12.5).

Proof. The theorem follows from Lemma 9.11 and Theorem 11.2, see also Lemma 11.4. □

12.3. **Asymptotics of solutions of Bethe ansatz equations.** Let $z = (z_1, \dots, z_{2m}) \in (\Lambda')^{2m}/S_{2m}$ have distinct coordinates. By Lemma 11.4 the Bethe ansats equations (6.2) extend to a holomorphic system of equations for $(\mu, t) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{C}^m$ at $\mu = \infty$.

Lemma 12.3. When $\mu = \infty$, a point $(t_1, \dots, t_m) \in \Lambda^m/S_m$ is a solution of (6.2) if and only if $\{t_1, \dots, t_m\}$ is a subset of size m of $\{z_1, \dots, z_{2m}\}$. □

Thus, solutions to (6.2) at $\mu = \infty$ for $t \in \Lambda^m/S_m$ are in bijection with m-element subsets $I \subset \{1, \dots, 2m\}$. By the implicit function theorem, each of these extends to a holomorphic family of solution $t_I(\mu) \in \Lambda^m/S_m$ to (6.2) as μ ranges through a neighborhood of ∞, see the proof of Theorem 11.2.

Thus, $t_I(\mu)$ is a holomorphic function of μ as $\mu \to \infty$ and has the property:

$$t_I(\mu) \to (z_j)_{j \in I}, \quad \text{as} \quad \mu \to \infty. \tag{12.9}$$

Lemma 12.4. For $I = \{i_1, \dots, i_m\}$, the coordinates $t_1(\mu), \dots, t_m(\mu)$ of the solution $t_I(\mu)$ can be ordered so that for $\mu \to \infty$ we have

$$t_j(\mu) = z_{i_j} + \frac{1}{2\pi i \mu} + O(\mu^{-2}), \tag{12.10}$$

where $j = 1, \dots, m$.

Proof. Denote $v = (2\pi i \mu)^{-1}$ as in Lemma 11.4. Introduce new variables $w_1, \dots, w_m$ by the formula $t_j = z_{i_j} + v w_j$. Then equations (6.2) can be written in the form

$$\frac{1}{v} - \rho(v w_j) + R_j(v, w_1, \dots, w_m) = 0, \qquad j = 1, \dots, m, \tag{12.11}$$

where $R_j(v, w_1, \dots, w_m)$ is a holomorphic function of its arguments at $v = 0$. Since $\rho(u) = u^{-1} + O(u)$ as $u \to 0$, equation (12.11) can be rewritten in the form $\frac{1}{v} - \frac{1}{v w_j} + \tilde{R}_j(v, w_1, \dots, w_m) = 0$, where $\tilde{R}_j(v, w_1, \dots, w_m)$ is another holomorphic function. Multiplying both sides of this equation by $v w_j$ we obtain an equation $w_j = 1 - v \tilde{R}_j(v, w_1, \dots, w_m)$, which implies that $w_j = 1 + O(v)$. This proves (12.10). □

As $\mu \to \infty$, the analytic involution $\beta^{-1} \circ \iota_{\mathrm{an}} \circ \beta$ sends the equivalence class of the solution $(\mu, t_I(\mu))$ to the equivalence class of the solution $(-\mu, t_{\bar{I}}(-\mu))$, where $\bar{I} = \{1, \dots, 2m\} - I$, see Lemma 11.4 and Theorem 12.2.

Corollary 12.5. For $\bar{I} = \{\bar{i}_1, \dots, \bar{i}_m\}$, the coordinates $t_1(-\mu), \dots, t_m(-\mu)$ of the solution $t_{\bar{I}}(-\mu)$ can be ordered so that for $\mu \to \infty$ we have

$$t_j(-\mu) = z_{\bar{i}_j} - \frac{1}{2\pi i \mu} + O(\mu^{-2}), \tag{12.12}$$

where $j = 1, \dots, m$. □

12.4. **Asymptotics of Bethe eigenfunctions.** Let $\mu \to \infty$. Choose one of the $\binom{2m}{m}$ families of Bethe ansatz solutions $t_I(\mu) \in \Lambda^m/S_m$, where $I = \{i_1, \dots, i_m\}$ is an m-element subset of $\{1, \dots, 2m\}$. Let us order the coordinates $t_1(\mu), \dots, t_m(\mu)$ so that the asymptotics (12.10) hold. Consider the eigenfunction $\Psi(\lambda_{12}, \mu, t_I(\mu), z)$ corresponding to this solution. Then the function

$$\Psi_I(\lambda_{12}, \mu) := \Psi(\lambda_{12}, \mu, t_I(\mu), z) \prod_{j=1}^{m} \theta(t_j(\mu) - z_{i_j}) \tag{12.13}$$

is also an eigenfunction of the dynamical elliptic Bethe algebra, since the last product is constant with respect to λ_{12}. The function $\Psi_I(\lambda_{12}, \mu)$ will be called the *normalized eigenfunction* corresponding to the eigenfunction $\Psi(\lambda_{12}, \mu, t_I(\mu), z)$.

Lemma 12.6. The normalized function $\Psi_I(\lambda_{12}, \mu)$ has an expansion of the form

$$\Psi_I(\lambda_{12}, \mu) = e^{\pi\mu\lambda_{12}} \sum_{k=0}^{\infty} w_k(\lambda_{12})\, \mu^{-k}, \tag{12.14}$$

where $w_0 = (-1)^m v_{\bar{I}}$ does not depend on λ_{12}, all the coefficients $w_k(\lambda_{12})$ are meromorphic functions of λ_{12} with poses at most at the points of the subset $\mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$, and the sum is uniformly convergent on any compact subset of $\mathbb{C} - (\mathbb{Z} + \tau\mathbb{Z})$ in the λ_{12}-line.

Proof. Clearly the function $F(\lambda_{12}, \mu) = e^{-\pi\mu\lambda_{12}}\theta(\lambda)^m \Psi_I(\lambda_{12}, \mu)$ is holomorphic in (λ_{12}, μ) in a neighborhood of the line $\mu = \infty$, moreover, $F(\lambda_{12}, \infty) = (-1)^m \theta(\lambda)^m v_{\bar{I}}$, see formula (10.2). This implies the lemma. □

Consider the eigenvalues $E_{0,I}(\mu), \dots, E_{2m,I}(\mu)$ of the dynamical Hamiltonians $H_0(z), \dots, H_{2m}(z)$ on the eigenfunction $\Psi_I(\lambda_{12}, \mu)$.

Lemma 12.7. As $\mu \to \infty$ we have

$$\begin{aligned} E_{0,I}(\mu) &= \frac{\pi i}{2}\mu^2 + \sum_{k=0}^{\infty} E_{0,I}^k \mu^{-k} \\ E_{a,I}(\mu) &= \pi i \mu + \sum_{k=0}^{\infty} E_{a,I}^k \mu^{-k}, \qquad a \in I, \\ E_{a,I}(\mu) &= -\pi i \mu + \sum_{k=0}^{\infty} E_{a,I}^k \mu^{-k}, \qquad a \notin I, \end{aligned} \tag{12.15}$$

where for $a = 0, \dots, 2m$, the functions $\sum_{k=0}^{\infty} E_{a,I}^k \mu^{-k}$ are holomorphic functions at $\mu = \infty$.

Proof. The eigenvalues of the KZB operators on the eigenfunction $\Psi(\lambda_{12}, \mu, t_I(\mu), z)$ were calculated in Section 5.3. Now the lemma follows from formula (12.10). □

Corollary 12.8. The formal series $e^{\pi\mu\lambda_{12}}\sum_{k=0}^{\infty} w_k(\lambda_{12})\,\mu^{-k}$ with respect to the variable μ is a solution of the equations

$$\begin{aligned} H_0(z)\sum_{k=0}^{\infty} w_k(\lambda_{12})\,\mu^{-k} &= \Big(\frac{\pi i}{2}\mu^2 + \sum\nolimits_{k=0}^{\infty} E_{0,I}^k\mu^{-k}\Big)\sum_{k=0}^{\infty} w_k(\lambda_{12})\,\mu^{-k}, \qquad (12.16)\\ H_a(z)\sum_{k=0}^{\infty} w_k(\lambda_{12})\,\mu^{-k} &= \Big(\pi i\mu + \sum\nolimits_{k=0}^{\infty} E_{a,I}^k\mu^{-k}\Big)\sum_{k=0}^{\infty} w_k(\lambda_{12})\,\mu^{-k}, \quad a\in I,\\ H_a(z)\sum_{k=0}^{\infty} w_k(\lambda_{12})\,\mu^{-k} &= \Big(-\pi i\mu + \sum\nolimits_{k=0}^{\infty} E_{a,I}^k\mu^{-k}\Big)\sum_{k=0}^{\infty} w_k(\lambda_{12})\,\mu^{-k}, \quad a\notin I. \end{aligned}$$

□

12.5. **Asymptotic eigenfunctions.** Consider a formal series

$$\Psi^{\mathrm{asy}}(\lambda_{12},\mu) = e^{2\pi i\mu\lambda_{12}}\sum_{k=0}^{\infty} u_k(\lambda_{12})\,\mu^{-k}, \qquad \mu\to\infty$$

with some coefficients $u_k(\lambda_{12})$.

Lemma 12.9. If there exists a formal series solution

$$\Psi^{\mathrm{asy}}(\lambda_{12},\mu) = e^{2\pi i\mu\lambda_{12}}\sum\nolimits_{k=0}^{\infty} u_k(\lambda_{12})\,\mu^{-k}$$

to the equations

$$\begin{aligned} H_0(z)\Psi^{\mathrm{asy}}(\lambda_{12},\mu) &= \Big(\frac{\pi i}{2}\mu^2 + \sum\nolimits_{k=0}^{\infty} E_{0,I}^k\mu^{-k}\Big)\Psi^{\mathrm{asy}}(\lambda_{12},\mu), \qquad (12.17)\\ H_a(z)\Psi^{\mathrm{asy}}(\lambda_{12},\mu) &= \Big(\pi i\mu + \sum\nolimits_{k=0}^{\infty} E_{a,I}^k\mu^{-k}\Big)\Psi^{\mathrm{asy}}(\lambda_{12},\mu), \quad a\in I,\\ H_a(z)\Psi^{\mathrm{asy}}(\lambda_{12},\mu) &= \Big(-\pi i\mu + \sum\nolimits_{k=0}^{\infty} E_{a,I}^k\mu^{-k}\Big)\Psi^{\mathrm{asy}}(\lambda_{12},\mu), \quad a\notin I, \end{aligned}$$

then it is unique up to multiplication by a scalar of the form $\sum_{k=0}^{\infty} c_k\mu^{-k}$, where $c_i\in\mathbb{C}$.

Proof. Equations (12.17) say that

$$\begin{aligned} u_0' &= 0, \qquad (12.18)\\ ((e_{22}-e_{11})^{(a)}-1)u_0 &= 0, \qquad a\in I,\\ ((e_{22}-e_{11})^{(a)}+1)u_0 &= 0, \qquad a\in \bar I, \end{aligned}$$

where u' is the derivative of the function u of the argument λ_{12} with respect to λ_{12}. Hence u_0 is a constant multiple of the vector $v_{\bar I}$, that is, $u_0 = c_0 v_{\bar I}$ for some $c_0\in\mathbb{C}$. For $k>0$, equations (12.17) say that

$$\begin{aligned} u_k' &= R_k^0(u_0,\dots,u_{k-1}), \qquad (12.19)\\ ((e_{22}-e_{11})^{(a)}-1)u_k &= R_k^a(u_0,\dots,u_{k-1}), \qquad a\in I,\\ ((e_{22}-e_{11})+1)u_k &= R_k^a(u_0,\dots,u_{k-1}), \qquad a\in \bar I, \end{aligned}$$

where $R_k^a(u_0,\,\dots,\,u_{k-1})$, for $a=0,\dots,2m$, are some explicit expressions in terms of the functions $u_0,\,\dots,\,u_{k-1}$. So if the coefficient u_k can be determined from the system (12.19), then it is unique up to addition of a constant multiple of $v_{\bar I}$. This proves the lemma. □

12.6. **Proof of Theorem 10.3.** Let $I \subset \{1,\dots,2m\}$ be an m-element subset and $\mu \to \infty$. Consider the solutions $(\mu, t_I(\mu))$ and $(-\mu, t_{\bar I}(-\mu))$ of the Bethe ansatz equations (6.2). They are related by the analytic involution $\beta^{-1}\circ\iota_{\rm an}\circ\beta$, see Section 12.3. Consider the associated eigenfunctions $\Psi(\lambda_{12},\mu,t_I(\mu),z)$ and $\Psi(\lambda_{12},-\mu,t_{\bar I}(-\mu),z)$ and their respective normalized versions, which we will denote by Ψ_I and $\Psi_{\bar I}$, respectively, see (12.13). Both have asymptotic expansions of Lemma 12.6,

$$\Psi_I = e^{\pi\mu\lambda_{12}}\sum\nolimits_{k=0}^{\infty} w_k^I(\lambda_{12})\,\mu^{-k}, \qquad \Psi_{\bar I} = e^{-\pi\mu\lambda_{12}}\sum\nolimits_{k=0}^{\infty} \bar w_k^{\bar I}(\lambda_{12})\,\mu^{-k},$$

where $w_0^I = (-1)^m v_{\bar I}$ and $\bar w_0^{\bar I} = (-1)^m v_I$. Recall the nontrivial element s of the $\mathfrak{sl}_2$ Weyl group and consider the third eigenfunction $s(\Psi_I)$. Its asymptotic expansion has the form $e^{-\pi\mu\lambda_{12}}\sum_{k=0}^{\infty} s.w_k^I(-\lambda_{12})\,\mu^{-k}$, where $s.w_0^I(-\lambda_{12}) = (-1)^m v_I$. By Lemma 12.9 we conclude that $s(\Psi_I) = \Psi_{\bar I}$. Hence Theorem 10.3 is proved for solutions of the form $(\mu, t_I(\mu))$. By Theorem 11.2 such solutions correspond to an open subset of the space Sol_m, which is irreducible. This proves Theorem 11.2 in full generality. □

References

[BMV] L. Borisov, E. Mukhin, A. Varchenko, unpublished, 2004

[FV1] G. Felder and A. Varchenko, *Integral representation of solutions of the elliptic Knizhnik-Zamolodchikov-Bernard equations*, Int. Math. Res. Notices, 1995, N. 5, 221–233

[FV2] G. Felder and A. Varchenko, *The elliptic gamma function and $SL(3,Z)\ltimes\mathbb{Z}^3$*, Adv. Math. **156** (2000), 44–76

[FW] G. Felder and C. Wieszerkowski, *Conformal blocks on elliptic curves and the KZB equations*, CMP **176** (1996), n.1, 133–161

[HU] R. Howe, T. Umeda, *The Capelli identity, the double commutant theorem, and multiplicity-free actions*, Math. Ann. **290** (1991), no. 3, 565–619

[JV] E. Jensen, A. Varchenko, *Norms of eigenfunctions to trigonometric KZB operators*, IMRN **6** (2013), 1230–1267, doi:10.1093/imrn/rns017

[MNO] A. Molev, M. Nazarov, G. Olshanski, *Yangians and classical Lie algebras*, Russian Math. Surveys **51** (1996), no. 2, 205–282

[MV1] E. Mukhin and A. Varchenko, *Critical points of master functions and flag varieties*, Commun. Contemp. Math. **6** (2004), no. 1, 111–163

[MV2] E. Mukhin and A. Varchenko, *Quasi-polynomials and the Bethe Ansatz*, Geometry & Topology Monographs **13**, Groups, homotopy and configuration spaces (Tokyo 2005), in honor of Fred Cohen, (2008), 385–420

[MTV1] E. Mukhin, V. Tarasov, and A. Varchenko, *The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz*, Ann. of Math. (2) **170** (2009), no. 2, 863–881

[MTV2] E. Mukhin, V. Tarasov, A. Varchenko, *Schubert calculus and representations of general linear group*, J. Amer. Math. Soc. **22** (2009), no. 4, 909–940

[RV] N. Reshetikhin and A. Varchenko, *Quasiclassical asymptotics of solutions to the KZ equations*, Geometry, Topology and Physics for R. Bott, Intern. Press, 1995, 293–322

[RST] V. Rubtsov, A. Silantyev, D. Talalaev, *Manin Matrices, Quantum Elliptic Commutative Families and Characteristic Polynomial of Elliptic Gaudin Model*, SIGMA **5** (2009), 110, 1–22

[ScV] I. Scherbak and A. Varchenko, *Critical points of functions, $\mathfrak{sl}_2$ representations, and Fuchsian differential equations with only univalued solutions*, Moscow Math. J., **3**, n. 2 (2003), 621–645

[WW] E.T. Whittaker and G.N. Watson, *A course of Modern Analysis*, Cambridge University Press, 1927

°,⋆ Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3250, USA

E-mail address: dthomp@email.unc.edu; anv@email.unc.edu

⋆Faculty of Mathematics and Mechanics, Lomonosov Moscow State University, Leninskiye Gory 1, 119991 Moscow GSP-1, Russia