

## DECOMPOSITIONS OF SIGNED DEFICIENT TOPOLOGICAL MEASURES

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**ABSTRACT.** This paper focuses on various decompositions of topological measures, deficient topological measures, signed topological measures, and signed deficient topological measures. These set functions generalize measures and correspond to certain nonlinear functionals. They may assume  $\infty$  or  $-\infty$ . We introduce the concept of a proper signed deficient topological measure and show that a signed deficient topological measure can be represented as a sum of a signed Radon measure and a proper signed deficient topological measure. We also generalize practically all known results that involve proper deficient topological measures and proper topological measures on compact spaces to locally compact spaces. We prove that the sum of two proper (deficient) topological measures is a proper (deficient) topological measure. We give a criterion for a (deficient) topological measure to be proper.

### 1. INTRODUCTION

Topological measures (initially called quasi-measures) were introduced by J. F. Aarnes in [1], [2], and [3]. These generalizations of measures and corresponding generalizations of linear functionals are connected to the problem of linearity of the expectational functional on the algebra of observables in quantum mechanics. Applications of topological measures and corresponding nonlinear functionals to symplectic topology have been studied in numerous papers beginning with [8] (which has been cited over 100 times), and in a monograph [15].

Topological measures are defined on open and closed subsets of a topological space, which means that there is no algebraic structure on the domain. They lack subadditivity and other properties typical for measures, and many standard

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techniques often employed for measures and linear functionals are no longer applicable to them. Nevertheless, many properties of measures still hold for topological measures.

The natural generalizations of topological measures are signed topological measures and deficient topological measures. Signed topological measures of finite norm on a compact space were introduced in [10] then studied and used in various works, including [11], [14], [17], and [19]. Deficient topological measures (as real-valued functions on a compact space) were first defined and used by A. Rustad and Ø. Johansen in [14] and later independently reintroduced and further developed by M. Svistula in [17] and [18]. In [6] we define and study signed deficient topological measures on locally compact spaces. Signed deficient topological measures generalize topological measures, signed topological measures, and deficient topological measures. These set functions may assume  $\infty$  or  $-\infty$ .

In this paper we introduce some new concepts, for example, proper signed deficient topological measures, and prove new results (including Theorem 3.9 and Theorem 4.1).

We also generalize practically all known results that involve proper deficient topological measures and proper topological measures on compact spaces to locally compact spaces. Also, we no longer require that all set functions have a finite “norm”. We prove that the sum of two proper (deficient) topological measures is a proper (deficient) topological measure. We give a criterion for a (deficient) topological measure to be proper.

The focus of this paper is various decompositions of topological measures, deficient topological measures, signed topological measures, and signed deficient topological measures. For example, we show that if  $\nu$  and  $\mu$  are signed topological measures, and at least one of them has a finite norm, then  $\nu - \mu$  is a signed topological measure; and if  $\mu \leq \nu$  then  $\nu - \mu$  is a topological measure. In many decompositions we use Radon measures and proper signed deficient topological measures. For example, a signed deficient topological measure can be represented as the sum of a signed Radon measure and a proper signed deficient topological measure. For a deficient topological measure  $\nu$ , the maximal Radon measure majorized by  $\nu$  is compact-finite if and only if  $\nu$  is singleton-finite.

In this paper  $X$  is a locally compact space. By  $\mathcal{O}(X)$  we denote the collection of open subsets of  $X$ ; by  $\mathcal{C}(X)$  the collection of closed subsets of  $X$ ; by  $\mathcal{K}(X)$  the collection of compact subsets of  $X$ . We denote by  $\overline{E}$  the closure of a set  $E$ , and  $\bigsqcup$  stands for a union of disjoint sets. We say that a signed set function is real-valued if its range is in  $\mathbb{R}$ . When we consider set functions into extended real numbers they are not identically  $\infty$  or  $-\infty$ .

## 2. PRELIMINARIES

**Definition 2.1.** Let  $X$  be a topological space and  $\nu$  be a set function on a family  $\mathcal{E}$  of subsets of  $X$  that contains  $\mathcal{O}(X) \cup \mathcal{C}(X)$ , and with values in  $[-\infty, \infty]$ . We say that

- $\nu$  is compact-finite if  $|\nu(K)| < \infty$  for any  $K \in \mathcal{K}(X)$ ;
- $\nu$  is singleton-finite if  $|\nu(\{x\})| < \infty$  for any  $x \in X$ ;

- $\nu$  is locally finite if every point  $x$  has a neighborhood  $U_x$  such that  $|\nu(U_x)| < \infty$ ;
- $\nu$  is simple if it only assumes values 0 and 1;
- $\nu$  is monotone if  $A \subseteq B$  implies  $\nu(A) \leq \nu(B)$ ;
- $\nu$  is inner regular (or inner compact regular) on  $A$  if  $\nu(A) = \lim\{\nu(K) : K \in \mathcal{K}(X), K \subseteq A\}$ ;
- a nonnegative monotone  $\nu$  is inner regular on  $A$  if  $\nu(A) = \sup\{\nu(K) : K \in \mathcal{K}(X), K \subseteq A\}$ ;
- $\nu$  is outer regular on  $A$  if  $\nu(A) = \lim\{\nu(U) : U \in \mathcal{O}(X), A \subseteq U\}$ ;
- a nonnegative monotone  $\nu$  is outer regular on  $A$  if  $\nu(A) = \inf\{\nu(U) : U \in \mathcal{O}(X), A \subseteq U\}$ ;
- a nonnegative monotone  $\nu$  is inner closed regular on  $A$  if  $\nu(A) = \sup\{\nu(F) : F \in \mathcal{C}(X), F \subseteq A\}$ ;
- $\nu$  is inner regular (outer regular, inner closed regular) if  $\nu$  is inner regular (outer regular, inner closed regular) on all sets  $A \in \mathcal{E}$ ;
- a nonnegative set function  $\nu$  is finite if  $\nu(X) < \infty$ .

*Remark 2.2.* For inner regularity on  $A$ , we mean the limit of the net  $\nu(K)$  with the index set  $\{K \in \mathcal{K}(X) : K \subseteq A\}$  ordered by inclusion. The limit exists and is equal to  $\nu(A)$ . Outer regularity on  $A$  is interpreted in a similar way, with the index set being  $\{U \in \mathcal{O}(X) : U \supseteq A\}$  ordered by reverse inclusion.

**Definition 2.3.** For a topological space  $X$ , by Borel sets we mean the members of the Borel  $\sigma$ -algebra on  $X$ . A Borel measure on  $X$  is a countably additive measure on the Borel  $\sigma$ -algebra on  $X$ .

A Radon measure  $m$  on  $X$  is a Borel measure that is compact-finite, outer regular on all Borel sets, and inner regular on all open sets, i.e.  $m(E) = \inf\{m(U) : E \subseteq U, U \text{ open}\}$  for every Borel set  $E$ , and  $m(U) = \sup\{m(K) : K \subseteq U, K \text{ compact}\}$  for every open set  $U$ . By a signed Radon measure we mean a difference of two Radon measures, at least one of which is finite.

Dealing with set functions on  $\mathcal{O}(X) \cup \mathcal{C}(X)$  in this paper, we may use the term “measure” for the restriction of a measure to  $\mathcal{O}(X) \cup \mathcal{C}(X)$ .

**Definition 2.4.** A deficient topological measure on a locally compact space  $X$  is a set function  $\nu : \mathcal{O}(X) \cup \mathcal{C}(X) \rightarrow [0, \infty]$  which is finitely additive on compact sets, inner regular on open sets, and outer regular on closed sets, i.e. :

- (DTM1) if  $C \cap K = \emptyset$ ,  $C, K \in \mathcal{K}(X)$  then  $\nu(C \sqcup K) = \nu(C) + \nu(K)$ ;
- (DTM2)  $\nu(U) = \sup\{\nu(C) : C \subseteq U, C \in \mathcal{K}(X)\}$  for  $U \in \mathcal{O}(X)$ ;
- (DTM3)  $\nu(F) = \inf\{\nu(U) : F \subseteq U, U \in \mathcal{O}(X)\}$  for  $F \in \mathcal{C}(X)$ .

Obviously, for a closed set  $F$ ,  $\nu(F) = \infty$  iff  $\nu(U) = \infty$  for every open set  $U$  containing  $F$ .

*Remark 2.5.* In [4, Section 3] it is proved that a deficient topological measure  $\nu$  is superadditive, i.e. if  $\bigsqcup_{t \in T} A_t \subseteq A$ , where  $A_t, A \in \mathcal{O}(X) \cup \mathcal{C}(X)$ , and at most one of the closed sets (if there are any) is not compact, then  $\nu(A) \geq \sum_{t \in T} \nu(A_t)$ . In particular,  $\nu$  is monotone, i.e. if  $A \subseteq B$ ,  $A, B \in \mathcal{O}(X) \cup \mathcal{C}(X)$  then  $\nu(A) \leq \nu(B)$ .

**Definition 2.6.** A topological measure on  $X$  is a set function  $\mu : \mathcal{O}(X) \cup \mathcal{C}(X) \rightarrow [0, \infty]$  satisfying the following conditions:

- (TM1) if  $A, B, A \sqcup B \in \mathcal{H}(X) \cup \mathcal{O}(X)$  then  $\mu(A \sqcup B) = \mu(A) + \mu(B)$ ;
- (TM2)  $\mu(U) = \sup\{\mu(C) : C \in \mathcal{H}(X), C \subseteq U\}$  for  $U \in \mathcal{O}(X)$ ;
- (TM3)  $\mu(F) = \inf\{\mu(U) : U \in \mathcal{O}(X), F \subseteq U\}$  for  $F \in \mathcal{C}(X)$ .

**Definition 2.7.** A signed deficient topological measure on a locally compact space  $X$  is a set function  $\nu : \mathcal{O}(X) \cup \mathcal{C}(X) \rightarrow [-\infty, \infty]$  that assumes at most one of  $\infty, -\infty$  and that is finitely additive on compact sets, inner regular on open sets, and outer regular on closed sets, i.e.

- (SDTM1) If  $C \cap K = \emptyset, C, K \in \mathcal{H}(X)$  then  $\nu(C \sqcup K) = \nu(C) + \nu(K)$ ;
- (SDTM2)  $\nu(U) = \lim\{\nu(K) : K \in \mathcal{H}(X), K \subseteq U\}$  for  $U \in \mathcal{O}(X)$ ;
- (SDTM3)  $\nu(F) = \lim\{\nu(U) : U \in \mathcal{O}(X), F \subseteq U\}$  for  $F \in \mathcal{C}(X)$ .

*Remark 2.8.* In condition **(SDTM2)**, we mean that the limit of the net  $\nu(K)$ , with the index set  $\{K \in \mathcal{H}(X) : K \subseteq U\}$  ordered by inclusion, exists and is equal to  $\nu(U)$ . Condition **(SDTM3)** is interpreted in a similar way, with the index set being  $\{U \in \mathcal{O}(X) : U \supseteq F\}$  ordered by reverse inclusion.

*Remark 2.9.* Since we consider set functions that are not identically  $\infty$  or  $-\infty$ , we see that for a signed deficient topological measure  $\nu(\emptyset) = 0$ . If  $\nu$  and  $\mu$  are signed deficient topological measures that agree on  $\mathcal{H}(X)$ , then  $\nu = \mu$ ; if  $\nu \leq \mu$  on  $\mathcal{H}(X)$  (or on  $\mathcal{O}(X)$ ), then  $\nu \leq \mu$ .

**Definition 2.10.** We define  $\|\nu\| = \sup\{|\nu(K)| : K \in \mathcal{H}(X)\}$  for a signed deficient topological measure  $\nu$ .

If  $\mu$  is a deficient topological measure then  $\|\mu\| = \mu(X)$ . For more information about  $\|\cdot\|$  for signed deficient topological measures see [6, Section 2].

**Definition 2.11.** A signed topological measure on a locally compact space  $X$  is a set function  $\mu : \mathcal{O}(X) \cup \mathcal{C}(X) \rightarrow [-\infty, \infty]$  that assumes at most one of  $\infty, -\infty$  and satisfies the following conditions:

- (STM1) if  $A, B, A \sqcup B \in \mathcal{H}(X) \cup \mathcal{O}(X)$  then  $\mu(A \sqcup B) = \mu(A) + \mu(B)$ ;
- (STM2)  $\mu(U) = \lim\{\mu(K) : K \in \mathcal{H}(X), K \subseteq U\}$  for  $U \in \mathcal{O}(X)$ ;
- (STM3)  $\mu(F) = \lim\{\mu(U) : U \in \mathcal{O}(X), F \subseteq U\}$  for  $F \in \mathcal{C}(X)$ .

By  $DTM(X), TM(X), SDTM(X)$ , and  $STM(X)$  we denote, respectively, the collections of all deficient topological measures on  $X$ , all topological measures on  $X$ , all signed deficient topological measures on  $X$ , and all signed topological measures on  $X$ .

*Remark 2.12.* Let  $X$  be locally compact. We denote by  $M(X)$  the collection of all Borel measures on  $X$  that are inner regular on open sets and outer regular (restricted to  $\mathcal{O}(X) \cup \mathcal{C}(X)$ ). Thus,  $M(X)$  includes regular Borel measures and Radon measures. Let  $SM(X)$  be the family of signed measures that are differences of two measures from  $M(X)$ , at least one of which is finite. In general (see [6, Section 3]),

$$M(X) \subsetneq TM(X) \subsetneq DTM(X), \text{ and } SM(X) \subsetneq STM(X) \subsetneq SDTM(X).$$

The following theorem (see [4, Theorem 32] gives criteria for a deficient topological measure to be a measure.

**Theorem 2.13.** *Let  $\mu$  be a deficient topological measure on a locally compact space  $X$ . The following are equivalent:*

- (a) *If  $C, K$  are compact subsets of  $X$ , then  $\mu(C \cup K) \leq \mu(C) + \mu(K)$ .*
- (b) *If  $U, V$  are open subsets of  $X$ , then  $\mu(U \cup V) \leq \mu(U) + \mu(V)$ .*
- (c)  *$\mu$  admits a unique extension to an inner regular on open sets, outer regular Borel measure  $m$  on the Borel  $\sigma$ -algebra of subsets of  $X$ .  $m$  is a Radon measure iff  $\mu$  is compact-finite. If  $\mu$  is finite then  $m$  is an outer regular and inner closed regular Borel measure.*

The following easy lemma can be found, for example, in [13, Chapter X, § 50, Theorem A].

**Lemma 2.14.** *If  $C \subseteq U \cup V$ , where  $C$  is compact and  $U, V$  are open, then there exist compact sets  $K$  and  $D$  such that  $C = K \cup D$ ,  $K \subseteq U$ ,  $D \subseteq V$ .*

The following fact is in [7, Chapter XI, 6.2]:

**Lemma 2.15.** *Let  $K \subseteq U$ ,  $K \in \mathcal{K}(X)$ ,  $U \in \mathcal{O}(X)$  in a locally compact space  $X$ . Then there exists a set  $V \in \mathcal{O}(X)$  such that  $K \subseteq V \subseteq \bar{V} \subseteq U$ ,  $\bar{V} \in \mathcal{K}(X)$ .*

The next definition is from [4, Section 2].

**Definition 2.16.** Given signed set function  $\lambda : \mathcal{K}(X) \rightarrow [-\infty, \infty]$  which assumes at most one of  $\infty, -\infty$  we define two set functions on  $\mathcal{O}(X) \cup \mathcal{C}(X)$ , the positive variation  $\lambda^+$  and the total variation  $|\lambda|$ , as follows: for an open subset  $U \subseteq X$  let

$$\lambda^+(U) = \sup\{\lambda(K) : K \subseteq U, K \in \mathcal{K}(X)\};$$

$$|\lambda|(U) = \sup\left\{\sum_{i=1}^n |\lambda(K_i)| : \bigsqcup_{i=1}^n K_i \subseteq U, K_i \in \mathcal{K}(X), n \in \mathbb{N}\right\};$$

and for a closed subset  $F \subseteq X$  let

$$\lambda^+(F) = \inf\{\lambda^+(U) : F \subseteq U, U \in \mathcal{O}(X)\};$$

$$|\lambda|(F) = \inf\{|\lambda|(U) : F \subseteq U, U \in \mathcal{O}(X)\}.$$

We define the negative variation  $\lambda^-$  associated with a signed set function  $\lambda$  as a set function  $\lambda^- = (-\lambda)^+$ .

*Remark 2.17.* If  $\lambda, \nu, \lambda \pm \nu$  are as in Definition 2.16, in [4, Sections 2 and 3] we show that  $\lambda \leq \lambda^+$  on  $\mathcal{K}(X)$ ; if  $\lambda \leq \nu$  on  $\mathcal{K}(X)$  then  $\lambda^+ \leq \nu^+$ ; also,  $|\lambda \pm \nu| \leq |\lambda| + |\nu|$ . If, in addition,  $\lambda$  is finitely additive on compact sets, then  $\lambda^+$  and  $|\lambda|$  are deficient topological measures. Thus, by Remark 2.5 they are monotone and superadditive. If  $\lambda$  is a deficient topological measure then  $\lambda^+ = \lambda$ . If  $\lambda$  is a signed inner regular measure, then  $\lambda^+$  and  $|\lambda|$  are the usual positive and total variations.

## 3. PROPER DEFICIENT TOPOLOGICAL MEASURES

*Remark 3.1.* If  $\nu$  is a nonnegative monotone set function that is outer regular on each singleton then  $\nu$  is singleton-finite iff  $\nu$  is locally finite. In particular, this holds for deficient topological measures. If  $\nu$  is also subadditive (in particular, if  $\nu$  is a measure) then  $\nu$  is locally finite iff  $\nu$  is compact-finite.

**Example 3.2.** Consider  $\mu$  on the Borel subsets of  $X$ . For a fixed  $x$  let  $\mu(A) = \infty$  if  $x \in A$ , and  $\mu(A) = 0$  otherwise. Then  $\mu$  is a Borel measure that is outer regular on all closed sets and that is not locally finite.

Here is an example of a deficient topological measure that is not singleton-finite. Fix  $x \in X$ , and define  $\lambda$  on  $\mathcal{K}(X)$  by:  $\lambda(K) = \infty$  if  $x \in K$  and  $\lambda(K) = 0$  if  $x \notin K$ . By Remark 2.17,  $\nu = \lambda^+$  is a deficient topological measure, and  $\nu$  is not singleton-finite.

**Definition 3.3.** A signed deficient topological measure  $\nu$  is called proper if from  $m \leq |\nu|$ , where  $m$  is a Radon measure and  $|\nu|$  is a total variation of  $\nu$ , it follows that  $m = 0$ .

*Remark 3.4.* If  $\nu$  is a deficient topological measure (in particular, a topological measure) then  $|\nu| = \nu$ , and  $\nu$  is a proper deficient topological measure if from  $m \leq \nu$ , where  $m$  is a Radon measure, it follows that  $m = 0$ . A signed deficient topological measure  $\nu$  is proper iff the deficient topological measure  $|\nu|$  is proper. A signed Radon measure  $m$  is proper iff  $|m|$  (which is a Radon measure) is proper, i.e. iff  $m = 0$ .

A finite Radon measure is an outer regular and inner closed regular Borel measure, so our definition of a proper deficient topological measure or a proper signed topological measure coincides with definitions in all previous papers, because they dealt with finite set functions on compact spaces.

**Definition 3.5.** Let  $X$  be locally compact. Suppose a set function  $\nu : \mathcal{K}(X) \rightarrow [0, \infty]$  is monotone, finitely additive, and  $\nu(\emptyset) < \infty$ . Define a set function  $\tilde{\nu} : \mathcal{K}(X) \rightarrow [0, \infty]$  by:

$$\tilde{\nu}(K) = \inf \left\{ \sum_{i=1}^n \nu(K_i) : K \subseteq \bigcup_{i=1}^n K_i, K_i \in \mathcal{K}(X), n \in \mathbb{N} \right\}.$$

In Definition 3.5 as  $\nu$  we may use a deficient topological measure, a topological measure or a measure. Also, if we consider set functions that are not identically  $\infty$ , monotone and finitely additive on compacts function  $\nu$  has  $\nu(\emptyset) = 0$ .

**Lemma 3.6.** *The following holds for  $\tilde{\nu}$ :*

- (p1)  $\tilde{\nu}(\emptyset) = 0$ .
- (p2)  $\tilde{\nu} \leq \nu$  on  $\mathcal{K}(X)$ .
- (p3)  $\tilde{\nu}$  is monotone, i.e.  $K \subseteq C$  implies  $\tilde{\nu}(K) \leq \tilde{\nu}(C)$ .
- (p4)  $\tilde{\nu}$  is subadditive, i.e.  $\tilde{\nu}(K \cup C) \leq \tilde{\nu}(K) + \tilde{\nu}(C)$ .
- (p5)  $\tilde{\nu}(K) = \inf \{ \sum_{i=1}^n \nu(C_i) : K = \bigcup_{i=1}^n C_i, C_i \in \mathcal{K}(X) \}$ .
- (p6)  $\tilde{\nu}$  is finitely additive on compact sets, i.e.  $\tilde{\nu}(C \sqcup K) = \tilde{\nu}(C) + \tilde{\nu}(K)$ .
- (p7) If  $\nu$  is compact-finite, then so is  $\tilde{\nu}$ .

*Proof.* Properties (p1) - (p3) easily follow from the definition of  $\tilde{\nu}$ .

(p4). If  $\tilde{\nu}(K) = \infty$  or  $\tilde{\nu}(C) = \infty$  then the subadditivity is obvious. Assume that  $\tilde{\nu}(K) < \infty$  and  $\tilde{\nu}(C) < \infty$ . For  $\epsilon > 0$  find compact sets  $K_i$ ,  $i = 1, \dots, n$  and  $C_j$ ,  $j = 1, \dots, m$  such that  $K \subseteq \bigcup_{i=1}^n K_i$ ,  $C \subseteq \bigcup_{j=1}^m C_j$ ,  $\sum_{i=1}^n \nu(K_i) - \nu(K) < \epsilon$ ,  $\sum_{j=1}^m \nu(C_j) - \nu(C) < \epsilon$ . Since the sets  $K_i, C_j$  cover  $C \cup K$ , we have  $\tilde{\nu}(K \cup C) \leq \sum_{i=1}^n \nu(K_i) + \sum_{j=1}^m \nu(C_j) < \tilde{\nu}(K) + \tilde{\nu}(C) + 2\epsilon$ . The subadditivity of  $\tilde{\nu}$  follows.

(p5). Let  $\alpha = \inf\{\sum_{i=1}^n \nu(C_i) : K = \bigcup_{i=1}^n C_i, C_i \in \mathcal{K}(X)\}$ . Clearly,  $\tilde{\nu}(K) \leq \alpha$ . On the other hand, for any cover  $\{K_i : i = 1, \dots, n\}$  of  $K$  we have  $K = \bigcup_{i=1}^n (K \cap K_i)$ , so  $\alpha \leq \sum_{i=1}^n \nu(K \cap K_i) \leq \sum_{i=1}^n \nu(K_i)$ , leading to  $\alpha \leq \tilde{\nu}(K)$ .

(p6). Suppose  $C \sqcup K \subseteq \bigcup_{i=1}^n C_i$  where  $C, K, C_i \in \mathcal{K}(X)$ . Note that by monotonicity and finite additivity of  $\nu$  on compact sets we have  $\nu(C_i) \geq \nu(C_i \cap C) + \nu(C_i \cap K)$ . Then  $\sum_{i=1}^n \nu(C_i) \geq \sum_{i=1}^n \nu(C_i \cap C) + \sum_{i=1}^n \nu(C_i \cap K) \geq \tilde{\nu}(C) + \tilde{\nu}(K)$ . It follows that  $\tilde{\nu}(C \sqcup K) \geq \tilde{\nu}(C) + \tilde{\nu}(K)$ . By part (p4)  $\tilde{\nu}(C \sqcup K) = \tilde{\nu}(C) + \tilde{\nu}(K)$ .

(p7). Follows from part (p2).  $\square$

**Lemma 3.7.** *Let  $X$  be locally compact. Suppose a set function  $\nu : \mathcal{K}(X) \rightarrow [0, \infty]$  is monotone, finitely additive, and  $\nu(\emptyset) < \infty$ . Let  $\tilde{\nu}^+$  be the positive variation according to Definition 2.16 of the set function  $\tilde{\nu}$  from Definition 3.5. Then*

- (i)  $\tilde{\nu}^+$  is a deficient topological measure.
- (ii) If  $\nu$  is a deficient topological measure then  $\tilde{\nu}^+ \leq \nu$ . If  $\nu$  is also compact-finite (respectively, finite) then so is  $\tilde{\nu}^+$ .
- (iii)  $\tilde{\nu}^+$  is finitely subadditive on  $\mathcal{O}(X)$ .
- (iv)  $m = \tilde{\nu}^+$  is (a restriction to  $\mathcal{O}(X) \cup \mathcal{C}(X)$  of) a Borel measure which is inner regular on open sets and outer regular. If  $\nu$  is compact-finite, then  $\tilde{\nu}^+$  is a Radon measure. If  $\nu$  is finite, then  $\tilde{\nu}^+$  is an outer regular and inner closed regular Borel measure.

*Proof.* (i). Follows from part (p6) of Lemma 3.6 and Remark 2.17.

(ii). By part (p2) of Lemma 3.6,  $\tilde{\nu} \leq \nu$  on  $\mathcal{K}(X)$ . By Remark 2.17  $\tilde{\nu}^+ \leq \nu^+ = \nu$ . The rest is obvious.

(iii). Let  $U, V$  be open and  $C \subseteq U \cup V$ , where  $C$  is compact. By Lemma 2.14,  $C = K \cup D$  where  $K, D \in \mathcal{K}(X)$ ,  $K \subseteq U, D \subseteq V$ . Since  $\tilde{\nu}$  is subadditive,  $\tilde{\nu}(C) \leq \tilde{\nu}(K) + \tilde{\nu}(D) \leq \tilde{\nu}^+(U) + \tilde{\nu}^+(V)$ . Taking supremum over compact sets  $C \subseteq U \cup V$  we obtain  $\tilde{\nu}^+(U \cup V) \leq \tilde{\nu}^+(U) + \tilde{\nu}^+(V)$ .

(iv). Follows from parts (iii), (ii) and Theorem 2.13.  $\square$

The next result gives the formula for  $m$  on  $\nu$ -finite open sets when  $\nu$  is a deficient topological measure.

**Proposition 3.8.** *Let  $X$  be locally compact, and  $\nu$  be a deficient topological measure on  $X$ . Then for any open set  $U$  with  $\nu(U) < \infty$*

$$\begin{aligned} m(U) = \tilde{\nu}^+(U) &= \inf \left\{ \sum_{i=1}^n \nu(A_i) : U \subseteq \bigcup_{i=1}^n A_i, A_i \in \mathcal{O}(X) \cup \mathcal{K}(X) \right\} \\ &= \inf \left\{ \sum_{i=1}^n \nu(U_i) : U \subseteq \bigcup_{i=1}^n U_i, U_i \in \mathcal{O}(X) \right\} \\ &= \inf \left\{ \sum_{i=1}^n \nu(U_i) : U = \bigcup_{i=1}^n U_i, U_i \in \mathcal{O}(X) \right\} \end{aligned}$$

*Proof.* Let  $\alpha, \beta, \gamma$  be the right hand side, respectively, of the first, second, and third line. Then  $\alpha \leq \beta \leq \gamma$ , where  $\gamma = \beta$  by an argument similar to the one for part (p5) of Lemma 3.6.

We shall show that  $\tilde{\nu}^+(U) \leq \alpha$ . Note that  $\alpha \leq \nu(U) < \infty$ . For  $\epsilon > 0$  pick sets  $A_i \in \mathcal{O}(X) \cup \mathcal{K}(X)$  such that  $U \subseteq \bigcup_{i=1}^n A_i$  and  $\sum_{i=1}^n \nu(A_i) < \alpha + \epsilon$ . By outer regularity we may assume that each  $A_i$  is open. Take any compact  $K \subseteq U$ . Using Lemma 2.14 write  $K = \bigcup_{i=1}^n K_i$ ,  $K_i \subseteq A_i$ . Then  $\tilde{\nu}(K) \leq \sum_{i=1}^n \nu(K_i) \leq \sum_{i=1}^n \nu(A_i) \leq \alpha + \epsilon$ . Taking supremum over  $K \subseteq U$  we obtain  $\tilde{\nu}^+(U) \leq \alpha + \epsilon$ , so  $\tilde{\nu}^+(U) \leq \alpha$ .

Now we shall show that  $\beta \leq \tilde{\nu}^+(U)$ . This inequality holds trivially if  $\tilde{\nu}^+(U) = \infty$ , so assume that  $\tilde{\nu}^+(U) < \infty$ . Let  $\epsilon > 0$ . Since  $\nu(U) < \infty$ , find  $K \subseteq U$  such that  $\nu(U) - \nu(K) < \epsilon$ , and then  $\nu(U \setminus K) < \nu(U) - \nu(K) < \epsilon$ . For  $K$  choose a cover  $K \subseteq \bigcup_{i=1}^n K_i$  such that  $\sum_{i=1}^n \nu(K_i) - \tilde{\nu}(K) < \epsilon$ . For each  $K_i$  choose  $U_i \in \mathcal{O}(X)$  such that  $K_i \subseteq U_i$  and  $\sum_{i=1}^n (\nu(U_i) - \nu(K_i)) < \epsilon$ . Then the sets  $\{U \setminus K, U_1, \dots, U_n\}$  cover  $U$ , and

$$\beta \leq \nu(U \setminus K) + \sum_{i=1}^n \nu(U_i) \leq \nu(U \setminus K) + \sum_{i=1}^n \nu(K_i) + \epsilon \leq \tilde{\nu}(K) + 3\epsilon \leq \tilde{\nu}^+(U) + 3\epsilon.$$

Thus,  $\beta \leq \tilde{\nu}^+(U)$ , and the proof is complete.  $\square$

**Theorem 3.9.** *Let  $\nu$  be a deficient topological measure, and  $m = \tilde{\nu}^+$  be the measure from Lemma 3.7. The following are equivalent:*

- (a)  $\nu$  is singleton-finite.
- (b)  $\tilde{\nu}$  is compact-finite.
- (c)  $m$  is compact-finite.

*Proof.* (a)  $\implies$  (b) Suppose  $\nu$  is singleton-finite. Let  $K \in \mathcal{K}(X)$ . For each  $x \in K$  let  $U_x \in \mathcal{O}(X)$  be such that  $x \in U_x$  and  $\nu(U_x) < \infty$ . By Lemma 2.15 let  $V_x \in \mathcal{O}(X)$  be such that  $x \in V_x \subseteq \overline{V_x} \subseteq U_x$  and  $\overline{V_x}$  is compact. Finitely many  $V_x$  cover  $K$ . So  $K \subseteq \bigcup_{i=1}^n V_{x_i} \subseteq \bigcup_{i=1}^n \overline{V_{x_i}}$ , and  $\sum_{i=1}^n \nu(\overline{V_{x_i}}) \leq \sum_{i=1}^n \nu(U_{x_i}) < \infty$ . Thus,  $\tilde{\nu}(K) < \infty$ .

(b)  $\implies$  (c) For an arbitrary compact  $C$  choose  $W \in \mathcal{O}(X)$  with compact closure such that  $C \subseteq W \subseteq \overline{W} \subseteq X$  and see that  $m(C) \leq m(W) = \sup\{\tilde{\nu}(D) : D \subseteq W, D \in \mathcal{K}(X)\} \leq \tilde{\nu}(\overline{W}) < \infty$ . Thus, the measure  $m$  is compact-finite.

(c)  $\implies$  (a) Since by Lemma 3.6  $0 \leq \tilde{\nu} \leq m$  on  $\mathcal{K}(X)$ , and  $m$  is compact-finite, we have  $0 \leq \tilde{\nu}(\{x\}) < \infty$ . By part (p5) of Lemma 3.6  $\nu(\{x\}) = \tilde{\nu}(\{x\}) < \infty$ .  $\square$



**Corollary 3.10.** *Let  $\nu$  be a singleton-finite deficient topological measure, and  $m = \tilde{\nu}^+$  be the measure from Lemma 3.7. Then  $m$  is the maximal of Radon measures that is majorized by  $\nu$ .*

*Proof.* By Theorem 3.9  $\tilde{\nu}^+$  is compact-finite, and by Theorem 2.13  $m$  is a Radon measure. Suppose  $\lambda$  is a Radon measure and  $\lambda \leq \nu$ . Take a compact  $K$ . If  $K \subseteq \bigcup_{i=1}^n K_i$ , where  $K_i \in \mathcal{K}(X)$ , then  $\lambda(K) \leq \sum_{i=1}^n \lambda(K_i) \leq \sum_{i=1}^n \nu(K_i)$ . So by Definition 3.5 and Remark 2.17  $\lambda(K) \leq \tilde{\nu}(K) \leq \tilde{\nu}^+(K)$ . By Remark 2.9  $\lambda \leq \tilde{\nu}^+$ .  $\square$

#### 4. DECOMPOSITIONS OF DEFICIENT TOPOLOGICAL MEASURES

**Theorem 4.1.** *Suppose  $\mu \leq \nu$ , where  $\mu$  is a signed topological measure, and  $\nu$  is a deficient topological measure. Suppose one of  $\mu, \nu$  is compact-finite and the other one is singleton-finite.*

- (i) *There exists a singleton-finite deficient topological measure  $\lambda$  such that  $\lambda = \nu - \mu$  on  $\mathcal{K}(X)$ .  $\lambda$  is compact-finite iff  $\mu$  and  $\nu$  are compact-finite.*
- (ii) *If  $\mu^-(X) < \infty$  (in particular, if  $0 \leq \mu \leq \nu$ ) then  $\nu = \mu + \lambda$ .*
- (iii) *If  $\|\mu\|, \|\nu\| < \infty$  then  $\lambda$  is finite.*

*Proof.* (i) We shall prove the statement in the case where  $\mu$  is compact-finite,  $\nu$  is singleton-finite; the other case is similar. Since  $\mu$  is real-valued on  $\mathcal{K}(X)$ , we may define  $\lambda$  as follows:

$$\begin{aligned} \lambda(K) &= \nu(K) - \mu(K) \text{ on } \mathcal{K}(X); \\ \lambda(U) &= \sup\{\lambda(K) : K \subseteq U, K \in \mathcal{K}(X)\} \text{ on } \mathcal{O}(X); \\ \lambda(F) &= \inf\{\lambda(U) : F \subseteq U, U \in \mathcal{O}(X)\} \text{ on } \mathcal{C}(X). \end{aligned}$$

Note that  $\lambda$  is a singleton-finite nonnegative set function. For  $K$  compact  $\lambda(K) < \infty$  iff  $\nu(K) < \infty$ , so  $\lambda$  is compact-finite iff  $\mu$  and  $\nu$  are.

First we shall show that  $\lambda$  is monotone on  $\mathcal{K}(X)$ . Suppose  $C, K \in \mathcal{K}(X)$  and  $C \subseteq K$ . If  $\nu(K) = \infty$ , then  $\lambda(K) = \infty \geq \lambda(C)$ . Now assume that  $\nu(K) < \infty$ . For  $\epsilon > 0$  there exists  $U \in \mathcal{O}(X)$  such that  $K \subseteq U$ ,  $\nu(U) - \nu(K) < \epsilon$  and  $|\mu(U) - \mu(K)| < \epsilon$ . Note that  $\nu(U), \nu(U \setminus C), \nu(C), \mu(U), \mu(C)$  are real numbers,  $\mu(U) = \mu(C) + \mu(U \setminus C)$ , and  $\nu(U) \geq \nu(U \setminus C) + \nu(C)$ . So we have  $\lambda(K) = \nu(K) - \mu(K) \geq \nu(U) - \epsilon - \mu(U) - \epsilon \geq \nu(U \setminus C) + \nu(C) - \mu(C) - \mu(U \setminus C) - 2\epsilon \geq \lambda(C) - 2\epsilon$ . Thus,  $\lambda(K) \geq \lambda(C)$ .

Now we shall show that this definition is consistent, i.e. if  $K$  is compact,  $\lambda(K) = \nu(K) - \mu(K)$ , and  $\lambda_1(K) = \inf\{\lambda(U) : K \subseteq U, U \in \mathcal{O}(X)\}$ , then  $\lambda(K) = \lambda_1(K)$ . For any  $U \in \mathcal{O}(X)$  containing  $K$  we have  $\lambda(U) \geq \lambda(K)$ , so  $\lambda_1(K) \geq \lambda(K)$ . If  $\nu(K) = \infty$ , then  $\lambda(K) = \infty$ , and then  $\lambda_1(K) = \infty = \lambda(K)$ . Now assume that  $\nu(K) < \infty$ . For  $\epsilon > 0$  choose  $U \in \mathcal{O}(X)$  such that  $K \subseteq U$ ,  $\nu(U) - \nu(K) < \epsilon$ , and  $|\mu(U) - \mu(K)| < \epsilon$ . It is not hard to see that  $\lambda(U) < \infty$ . We may choose a compact  $C$  such that  $K \subseteq C \subseteq U$ ,  $\lambda(U) - \lambda(C) < \epsilon$ , and  $|\mu(U) - \mu(C)| < \epsilon$ . Note that  $|\nu(C) - \nu(K)| \leq 2\epsilon$  and  $|\mu(C) - \mu(K)| \leq 2\epsilon$ , so  $|\lambda(C) - \lambda(K)| \leq 4\epsilon$ . Then

$$\lambda_1(K) \leq \lambda(U) \leq \lambda(C) + \epsilon \leq |\lambda(C) - \lambda(K)| + \lambda(K) + \epsilon \leq \lambda(K) + 5\epsilon.$$

It follows that  $\lambda_1(K) \leq \lambda(K)$ . Thus,  $\lambda(K) = \lambda_1(K)$ .

Finite additivity of  $\lambda$  on disjoint compact sets follows from the same property for  $\nu$  and  $\mu$ . Since outer regularity on closed sets and inner regularity on open sets is built in the definition of  $\lambda$ , we see that  $\lambda$  is a deficient topological measure.

(ii) If  $\mu^-(X) < \infty$  then  $\mu(A), \nu(A) \in (-\infty, \infty]$  for any  $A \in \mathcal{O}(X) \cup \mathcal{C}(X)$ . Let  $U$  be open. Checking all possible cases (namely,  $\mu(U), \nu(U) \in \mathbb{R}$ ;  $\mu(U) \in \mathbb{R}, \nu(U) = \infty$ ;  $\mu(U), \nu(U) = \infty$ ) we see that  $\nu(U) = \mu(U) + \lambda(U)$ . Then it is not hard to check that  $\nu(F) = \mu(F) + \lambda(F)$  for any  $F \in \mathcal{C}(X)$ . Thus,  $\nu = \mu + \lambda$ .

(iii) Easy to see. □

*Remark 4.2.* When  $X$  is compact,  $\mu \leq \nu$ , and both  $\mu$  and  $\nu$  are topological measures (necessarily finite), Wheeler [20, Proposition 3.4], proved that  $\nu - \mu$  is also a topological measure by considering corresponding quasi-linear functionals. In [17, Proposition 6] it was shown that if  $X$  is compact,  $\mu$  is a signed topological measure with finite norm, and  $\nu$  is a finite deficient topological measure, then  $\nu - \mu$  is a finite deficient topological measure. When  $X$  is locally compact, one can not prove Theorem 4.1 using the method from [17, Proposition 6]; it does not work even in the case when both set functions are topological measures if they are not both finite.

**Theorem 4.3.** *Let  $\nu$  be a singleton-finite deficient topological measure. There is the unique decomposition  $\nu = m + \nu'$ , where  $m$  is a Radon measure, and  $\nu'$  is a proper singleton-finite deficient topological measure. In fact,  $m$  is the maximal Radon measure majorized by  $\nu$ .  $\nu$  is compact-finite (respectively, finite) iff  $\nu'$  and  $m$  are.  $\nu$  is a finite topological measure iff  $\nu'$  is a finite proper topological measure and  $m$  is a finite Radon measure.  $\nu$  is a finite measure from  $M(X)$  iff  $\nu'$  is a finite proper measure from  $M(X)$  and  $m$  is a finite Radon measure.*

*Proof.* As in Corollary 3.10, let  $m = \tilde{\nu}^+$  be the maximal Radon measure majorized by  $\nu$ . By Theorem 4.1 there exists a singleton-finite deficient topological measure  $\nu'$  such that  $\nu = m + \nu'$ . Note that  $\nu'$  is proper, otherwise,  $m$  would not be the maximal Radon measure majorized by  $\nu$ .

We shall show that the decomposition is unique. Assume that  $\nu = m + \nu' = l + \lambda'$ , where  $m, l$  are Radon measures and  $\nu', \lambda'$  are proper singleton-finite deficient topological measures. Since  $m$  is the maximal of Radon measures majorized by  $\nu$ ,  $l \leq m$ . On  $\mathcal{H}(X)$  we have  $0 \leq m - l \leq \lambda'$ . Since  $m - l$  is a real-valued, finitely additive set function on  $\mathcal{H}(X)$ , we may obtain the deficient topological measure  $(m - l)^+$ , and by Remark 2.17  $0 \leq (m - l)^+ \leq (\lambda')^+ = \lambda'$ . Note that  $(m - l)^+(K) \leq m^+(K) = m(K) < \infty$ , so  $(m - l)^+$  is compact-finite. We claim that deficient topological measure  $(m - l)^+$  is subadditive on open sets. [Indeed, if  $C \subseteq U \cup V$ , where  $C \in \mathcal{H}(X), U, V \in \mathcal{O}(X)$ , by Lemma 2.14 write  $C = K \cup D$ , where  $K \subseteq U, D \subseteq V$ , and then  $(m - l)(C) = (m - l)(K \cup D) = m(K) + m(D) - m(K \cap D) - l(K) - l(D) + l(K \cap D) \leq (m - l)(K) + (m - l)(D) \leq (m - l)^+(U) + (m - l)^+(V)$ . Taking supremum over all compacts  $C \subseteq U \cup V$  we obtain  $(m - l)^+(U \cup V) \leq (m - l)^+(U) + (m - l)^+(V)$ .] By Theorem 2.13  $(m - l)^+$  is a Radon measure. Since  $(m - l)^+ \leq \lambda'$  and  $\lambda'$  is proper, we have  $(m - l)^+ = 0$ . For each compact  $K$  we have  $0 \leq (m - l)(K) \leq (m - l)^+(K) = 0$ ,

i.e.  $m(K) = l(K)$ . Then also  $\nu'(K) = \lambda'(K)$ . Thus,  $m = l$ ,  $\nu' = \lambda'$  and the decomposition of  $\nu$  is unique.

If  $\nu$  is compact-finite (respectively, finite), then by Lemma 3.7 so is  $m$ , and then by Theorem 4.1 so is  $\nu'$ .

If  $\nu$  is a finite topological measure (respectively, a finite measure from  $M(X)$ ) then  $m$  is a finite Radon measure, and it is easy to check that  $\nu' = \nu - m$  is a finite topological measure (respectively, a finite measure from  $M(X)$ ). The rest of the claims are easy.  $\square$

**Theorem 4.4.** (I) *A singleton-finite deficient topological measure  $\nu$  is proper iff given a compact  $K$  and  $\epsilon > 0$ , there are finitely many compact sets  $K_1, \dots, K_n$  such that  $K = \bigcup_{i=1}^n K_i$  and  $\sum_{i=1}^n \nu(K_i) < \epsilon$ .*  
 (II) *A finite deficient topological measure is proper iff given  $\epsilon > 0$ , there are finitely many sets  $A_1, \dots, A_n$  from  $\mathcal{O}(X)$  (or from  $\mathcal{O}(X) \cup \mathcal{K}(X)$ ) such that  $X = \bigcup_{i=1}^n A_i$  and  $\sum_{i=1}^n \nu(A_i) < \epsilon$ .*

*Proof.* By Theorem 4.3 a deficient topological measure  $\nu$  is proper iff  $m = \tilde{\nu}^+ = 0$ , where measure  $m$  is as in Corollary 3.10. (I). Then  $m(U) = 0$  for any open set  $U$ , i.e.  $\tilde{\nu}(K) = 0$  for any compact  $K$ . The statement now follows from part (p5) of Lemma 3.6.

(II). Then  $m(X) = \tilde{\nu}^+(X) = 0$ , and the statement now follows from Proposition 3.8.  $\square$

**Theorem 4.5.** *The sum of two proper singleton-finite deficient topological measures is a proper singleton-finite deficient topological measure.*

*Proof.* Let  $\nu, \mu$  be proper singleton-finite deficient topological measures. Clearly,  $\nu + \mu$  is a singleton-finite deficient topological measure. Suppose  $\lambda$  is a Radon measure and  $\lambda \leq \nu + \mu$ . We may consider  $\lambda$  on Borel subsets of  $X$ . Let  $K \in \mathcal{K}(X)$  and  $\epsilon > 0$ . By Theorem 4.4, let  $\{K_i\}_{i=1}^n$  and  $\{D_j\}_{j=1}^m$  be compact sets such that  $K = \bigcup_{i=1}^n K_i = \bigcup_{j=1}^m D_j$ ,  $\sum_{i=1}^n \nu(K_i) < \epsilon$ ,  $\sum_{j=1}^m \mu(D_j) < \epsilon$ . Set  $A_1 = K_1, A_2 = K_2 \setminus K_1, \dots, A_n = K_n \setminus \bigcup_{i=1}^{n-1} K_i$ . Similarly, let  $B_1 = D_1, B_2 = D_2 \setminus D_1, \dots, B_m = D_m \setminus \bigcup_{j=1}^{m-1} D_j$ . Then  $A_i \subseteq K_i, B_j \subseteq D_j$ , the sets  $A_i, B_j$  are disjoint, and  $K = \bigsqcup_{i=1}^n A_i = \bigsqcup_{j=1}^m B_j$ . Then  $K = \bigsqcup_{i,j} A_i \cap B_j$ ,  $A_i = \bigsqcup_{j=1}^m A_i \cap B_j$ ,  $B_j = \bigsqcup_{i=1}^n A_i \cap B_j$ . Note that  $\lambda(A_i \cap B_j) \leq \lambda(K_i \cap D_j) \leq \nu(K_i) + \mu(D_j) < \infty$ . Since  $\lambda$  is a Radon measure, it is inner regular on  $\sigma$ -finite sets (see, for example, [9, §7.2]). Thus, given  $\epsilon > 0$ , we may choose compact sets  $C_{ij} \subseteq A_i \cap B_j$  for  $i = 1, \dots, n, j = 1, \dots, m$  such that

$$\sum_{i,j} \lambda(C_{ij}) > \sum_{i,j} \lambda(A_i \cap B_j) - \epsilon = \lambda(K) - \epsilon.$$

Also,

$$\begin{aligned} \sum_{i,j} \lambda(C_{ij}) &\leq \sum_{i,j} \nu(C_{ij}) + \sum_{i,j} \mu(C_{ij}) = \sum_{i=1}^n \nu\left(\bigsqcup_{j=1}^m C_{ij}\right) + \sum_{j=1}^m \mu\left(\bigsqcup_{i=1}^n C_{ij}\right) \\ &\leq \sum_{i=1}^n \nu(K_i) + \sum_{j=1}^m \mu(D_j) < 2\epsilon. \end{aligned}$$

Then  $\lambda(K) < 3\epsilon$ . It follows that  $\lambda = 0$  on  $\mathcal{K}(X)$ . (This proof is adapted from [17, Theorem 3]). Therefore,  $\lambda = 0$ , and  $\nu + \mu$  is proper.  $\square$

*Remark 4.6.* If  $\nu$  is a proper deficient topological measure, then so is  $a\nu$  for any positive  $a$ . Using Theorem 4.5 we see that the set of all singleton-finite proper deficient topological measures is a positive cone.

**Theorem 4.7.** *Suppose  $\mu$  is a topological measure,  $\nu$  is a deficient topological measure, one of  $\mu, \nu$  is singleton-finite, and the other is compact-finite. Suppose  $\mu \leq \nu$ . As in Theorem 4.3, let  $\mu = m + \mu'$ ,  $\nu = n + \nu'$  be decompositions, where  $\mu', \nu'$  are proper singleton-finite deficient topological measures, and  $m, n$  are Radon measures. Then  $m \leq n$  and  $\mu' \leq \nu'$ .*

*Proof.* By Theorem 4.1 write  $\nu = \mu + \lambda$ , where  $\lambda$  is a singleton-finite deficient topological measure. By Theorem 4.3 we have  $\lambda = l + \lambda'$ , where  $l$  is a Radon measure and  $\lambda'$  is a proper singleton-finite deficient topological measure. Then  $\nu = \mu + \lambda = m + \mu' + l + \lambda' = (m + l) + (\mu' + \lambda')$ , where  $\mu' + \lambda'$  is a proper singleton-finite deficient topological measure by Theorem 4.5, and  $m + l$  is a Radon measure. By Theorem 4.3  $\nu = n + \nu'$  is the unique such decomposition. Then  $m + l = n$ ,  $\mu' + \lambda' = \nu'$  and so  $m \leq n$ ,  $\mu' \leq \nu'$ .  $\square$

**Corollary 4.8.** *Suppose  $\mu$  is a singleton-finite deficient topological measure,  $l$  is a Radon measure, and  $\mu \leq l$ . Then  $\mu$  is a Radon measure.*

*Proof.* By Theorem 4.3 write decompositions  $\mu = m + \mu'$ ,  $l = l + 0$ . By Theorem 4.7  $\mu' \leq 0$ , i.e.  $\mu' = 0$ . Then  $\mu = m$  is a Radon measure.  $\square$

**Definition 4.9.** A deficient topological measure is called extreme if it is an extreme point of the set of all deficient topological measures.

**Lemma 4.10.** *A finite extreme deficient topological measure is either a finite proper deficient topological measure or a finite Radon measure.*

*Proof.* Let  $\nu$  be a (nonzero) finite extreme deficient topological measure. By Theorem 4.3 we write  $\nu = m + \nu'$ , where  $m$  is a finite Radon measure and  $\nu'$  is a finite proper deficient topological measure. We may assume that  $m(X), \nu'(X) \neq 0$ , otherwise the statement follows. Then  $0 < m(X), \nu'(X) \leq \nu(X) < \infty$  and, following an argument from [16, Corollary 1], we may write  $\nu = a\nu_1 + b\nu_2$ , where  $a = \frac{m(X)}{\nu(X)}$ ,  $b = \frac{\nu'(X)}{\nu(X)}$ ,  $\nu_1 = \frac{\nu(X)}{m(X)}m$ ,  $\nu_2 = \frac{\nu(X)}{\nu'(X)}\nu'$ . But  $a + b = 1$  and  $\nu$  is extreme, so either  $a = 0$  and  $\nu = \nu'$ , or  $b = 0$  and  $\nu = m$ , and the statement follows.  $\square$

**Lemma 4.11.** *Let  $X$  be locally compact. If  $\mu$  is a simple deficient topological measure on  $X$  and there exists  $x$  such that  $\mu(\{x\}) = 1$  then  $\mu$  is a point mass at  $x$ .*

*Proof.* Let  $x \in X$  be such that  $\mu(\{x\}) = 1$ . Then by superadditivity of  $\mu$  we have  $\mu(X) \geq \mu(\{x\}) + \mu(X \setminus \{x\})$ , and it follows that  $\mu(X \setminus \{x\}) = 0$ . It is easy to see that  $\mu$  is finitely subadditive on compact sets. By Theorem 2.13  $\mu$  is a Borel measure on  $X$ , so it is a point mass at  $x$ .  $\square$

**Lemma 4.12.** *Let  $X$  be locally compact. The following are equivalent for a deficient topological measure  $\nu$  that assumes finitely many values:*

- (a)  $\nu$  is proper.
- (b) There are open sets  $U_1, \dots, U_n$  such that  $X = \bigcup_{i=1}^n U_i$  and  $\nu(U_i) = 0$ ,  $i = 1, \dots, n$ .
- (c)  $\nu(\{x\}) = 0$  for every  $x \in X$ .

*Proof.* By Theorem 4.4, (a) and (b) are equivalent. Clearly, (b) implies (c). We shall show that (c) implies (b). Let  $\mathcal{U}$  be a finite open cover of  $X$ , and suppose  $U \in \mathcal{U}$  is such that  $\nu(U) = a > 0$ . There exists  $K \in \mathcal{K}(X)$  such that  $K \subseteq U$  and  $\nu(K) = a$ . Since  $U = (U \setminus K) \sqcup K$  and by superadditivity  $\nu(U) \geq \nu(U \setminus K) + \nu(K)$ , it follows that  $\nu(U \setminus K) = 0$ . For each  $x \in K$  there is an open set  $V_x$  such that  $\nu(V_x) = 0$ . Finitely many of  $V_x$  cover  $K$ , and we call them  $V_1, \dots, V_n$ . So  $U \subseteq (U \setminus K) \cup V_1 \cup \dots \cup V_n$ , and  $\nu(U \setminus K) = \nu(V_1) = \dots = \nu(V_n) = 0$ . Replacing the set  $U$  in the cover  $\mathcal{U}$  by open sets  $U \setminus K, V_1, \dots, V_n$  and repeating the same procedure for each set  $U \in \mathcal{U}$  for which  $\nu(U) > 0$ , we obtain a new finite open cover  $\mathcal{W}$  with the property that  $\nu(W) = 0$  for each set  $W \in \mathcal{W}$ . Thus, (b) is satisfied.  $\square$

## 5. DECOMPOSITIONS OF SIGNED DEFICIENT TOPOLOGICAL MEASURES

**Theorem 5.1.** *Suppose  $\nu$  and  $\mu$  are signed topological measures, and at least one of  $\|\mu\|, \|\nu\|$  is finite. Then  $\lambda = \nu - \mu$  is a signed topological measure. If  $\mu \leq \nu$  then  $\lambda = \nu - \mu$  is a topological measure.*

*Proof.* Since a signed topological measure assumes at most one of  $\infty, -\infty$ , and at least one of  $\|\mu\|, \|\nu\|$  is finite, it is easy to check that  $\lambda = \nu - \mu$  is a signed topological measure. If  $\mu \leq \nu$  then  $\lambda = \nu - \mu \geq 0$ . Note that  $\lambda$  is monotone on  $\mathcal{K}(X)$ . [Indeed, if  $K \subseteq C$  then clearly  $\lambda(K) \leq \lambda(C)$  if  $\lambda(C) = \infty$ ; and if  $\lambda(C) < \infty$ , for  $\epsilon > 0$  choose  $U \in \mathcal{O}(X)$  such that  $C \subseteq U, |\lambda(U) - \lambda(C)| < \epsilon$ , and then  $\lambda(C) \geq \lambda(U) - \epsilon = \lambda(K) + \lambda(U \setminus K) - \epsilon \geq \lambda(K) - \epsilon$ , so  $\lambda(C) \geq \lambda(K)$ .] Then it is easy to check that  $\lambda$  is monotone. A signed topological measure  $\lambda \geq 0$  and monotone, so  $\lambda$  is a topological measure.  $\square$

*Remark 5.2.* If a signed deficient topological measure  $\mu$  is proper, then so is  $-\mu$ . If  $\mu, \nu$  are proper singleton-finite signed deficient topological measures such that  $\mu \pm \nu$  is defined and  $|\mu \pm \nu|$  is majorized by a sum of two proper singleton-finite deficient topological measures (which is a proper deficient topological measure) then  $\mu \pm \nu$  is a proper singleton-finite signed deficient topological measure. In particular, the family of all proper signed topological measures with finite norms is a linear space.

**Theorem 5.3.** *Suppose a signed singleton-finite deficient topological measure  $\mu$  is the difference of two deficient topological measures, one singleton-finite, and the other finite.*

- (i) *There is a decomposition  $\mu = m + \mu'$ , where  $m$  is a signed Radon measure, and  $\mu'$  is a proper singleton-finite signed deficient topological measure.*
- (ii) *If  $\mu$  is the difference of two finite deficient topological measures then the decomposition is unique, and  $\|m\|, \|\mu'\| < \infty$ .*
- (iii) *If  $\mu$  is a signed topological measure with  $\|\mu\| < \infty$  then the decomposition is unique,  $\mu'$  is a proper signed topological measure, and  $\|m\|, \|\mu'\| < \infty$ .*

*Proof.* (i). Assume  $\mu = \nu - \lambda$ , where  $\nu$  is singleton-finite and  $\lambda$  is finite (in the other case consider  $-\mu$ ). By Theorem 4.3 write  $\nu = n + \nu'$ ,  $\lambda = l + \lambda'$ , where  $n$  is a Radon measure,  $l$  is a finite Radon measure,  $\nu'$  is a singleton-finite proper deficient topological measure, and  $\lambda'$  is a finite proper deficient topological measure. Then  $\mu = (n - l) + (\nu' - \lambda')$ . Note that  $m = n - l$  is a signed Radon measure, and  $\mu' = \nu' - \lambda'$  is a singleton-finite signed deficient topological measure which is proper by Remark 5.2 (for  $|\nu' - \lambda'|$  is majorized by  $\nu' + \lambda'$ ).

(ii). Assume  $\mu = \nu - \lambda$ , where  $\nu$  and  $\lambda$  are finite. In the decomposition  $\mu = m + \mu'$  as in part (i) we see that  $\|m\| < \infty$  and  $\|\mu'\| < \infty$ . To show the uniqueness of the decomposition, assume that  $\mu = m + \mu' = m_1 + \mu'_1$ , where  $m, m_1$  are signed Radon measures,  $\mu', \mu'_1$  are proper signed deficient topological measures, and  $\|m\|, \|m_1\|, \|\mu'\|, \|\mu'_1\| < \infty$ . Then  $m - m_1 = \mu'_1 - \mu'$  is a finite signed Radon measure which is proper by Remark 5.2 (for  $|\mu'_1 - \mu'|$  is majorized by  $\mu'_1 + \mu'$ ). By Remark 3.4  $m - m_1 = \mu'_1 - \mu' = 0$ , and the decomposition is unique.

(iii). If  $\mu$  is a signed topological measure with  $\|\mu\| < \infty$ , by [6, Theorem 41] we may write  $\mu = \mu^+ - \mu^-$ , where  $\mu^+$  and  $\mu^-$  are finite deficient topological measures. Apply part (ii) and note that  $\mu' = \mu - m$  is a signed topological measure.  $\square$

**Theorem 5.4.** *Suppose  $\mu, \nu$  are singleton-finite signed topological measures, and  $\|\mu\| < \infty$ . Let  $\mu = m + \mu'$ , where  $m$  is a signed Radon measure,  $\mu'$  is a proper signed deficient topological measure, and  $\|m\|, \|\mu'\| < \infty$ , be the unique decomposition as in Theorem 5.3. Assume that  $\nu = n + \nu'$ , where  $n$  is a signed Radon measure and  $\nu'$  is a proper singleton-finite signed deficient topological measure, is the unique decomposition (see, for example, Theorem 5.3 or Theorem 4.3).*

- (i) *If  $\mu \leq \nu$  then  $m \leq n$  and  $\mu' \leq \nu'$ . If  $\nu \leq \mu$  then  $n \leq m$  and  $\nu' \leq \mu'$ .*
- (ii) *If  $|\mu(A)| \leq \nu(A)$  for any  $A \in \mathcal{K}(X)$  (or any  $A \in \mathcal{O}(X)$ ), then  $-n \leq m \leq n$ ,  $-\nu' \leq \mu' \leq \nu'$ .*

*Proof.* (i). Suppose first  $\mu \leq \nu$ . By Theorem 5.1 there is a topological measure  $\lambda = \nu - \mu$ , so  $\nu = \mu + \lambda$ . Note that  $\lambda$  is singleton-finite. By Theorem 4.3 we have  $\lambda = l + \lambda'$ , where  $l$  is a Radon measure and  $\lambda'$  is a proper singleton-finite deficient topological measure. Then  $\nu = (m + l) + (\mu' + \lambda')$ , where  $m + l$  is a signed Radon measure and  $\mu' + \lambda'$  is a proper singleton-finite signed deficient topological measure by Theorem 4.5. Since  $\nu = n + \nu'$  is the unique such decomposition, we have  $n = m + l$  and  $\nu' = \mu' + \lambda'$ . Then  $m \leq n$  and  $\mu' \leq \nu'$ .

Now suppose  $\nu \leq \mu$ . By Theorem 5.1 there is a topological measure  $\lambda = \mu - \nu$ , so  $\nu = \mu - \lambda$ , and  $\lambda$  is singleton-finite. As above,  $\nu = (m - l) + (\mu' - \lambda')$ , where  $m - l$  is a signed Radon measure and  $\mu' - \lambda'$  is a proper singleton-finite signed deficient topological measure by Remark 5.2 (for  $|\mu' - \lambda'|$  is majorized by  $\mu' + \lambda'$ ). Since  $\nu = n + \nu'$  is the unique such decomposition, we see that  $n = m - l$  and  $\nu' = \mu' - \lambda'$ . Then  $n \leq m$  and  $\nu' \leq \mu'$ .

(ii). If  $|\mu(A)| \leq \nu(A)$  for any  $A \in \mathcal{K}(X)$  (or any  $A \in \mathcal{O}(X)$ ) then by Remark 2.9  $-\nu \leq \mu \leq \nu$ , and so by part (i) we have  $-n \leq m \leq n$ ,  $-\nu' \leq \mu' \leq \nu'$ .  $\square$

*Remark 5.5.* The proof of Theorem 4.7 is similar to the one for Theorem 5.4. It uses Theorem 4.1 to get a deficient topological measure  $\lambda$  such that  $\nu = \mu + \lambda$ , and then uses decompositions of  $\mu$ ,  $\nu$ , and  $\lambda$  according to Theorem 4.3.

**Corollary 5.6.** *Suppose  $\mu$  is a signed singleton-finite topological measure,  $l$  is a finite Radon measure, and  $|\mu(A)| \leq l(A)$  for any  $A \in \mathcal{K}(X)$  (or any  $A \in \mathcal{O}(X)$ ). Then  $\mu$  is a signed Radon measure with finite norm.*

*Proof.* From Definition 2.10 we see that  $\|\mu\| \leq l(X) < \infty$ . By Theorem 5.3 we may write  $\mu = m + \mu'$ , where  $m$  is a signed Radon measure,  $\mu'$  is a proper signed topological measure, and  $\|m\|, \|\mu'\| < \infty$ . By Theorem 4.3 write the unique decomposition  $l = l + 0$ . By part (ii) of Theorem 5.4  $\mu' = 0$ , and so  $\mu = m$  is a signed Radon measure with finite norm.  $\square$

**Corollary 5.7.** *Suppose  $\mu$  is a signed singleton-finite topological measure. Then  $\mu$  is a signed Radon measure with finite norm iff  $|\mu|$  is a finite Radon measure.*

*Proof.* ( $\Leftarrow$ ) Since  $|\mu(K)| \leq |\mu|(K)$  on compact sets (see [4, Lemma 10]), by Corollary 5.6  $\mu$  is a signed Radon measure with finite norm. ( $\Rightarrow$ ) Clear.  $\square$

**Corollary 5.8.** *Suppose  $n$  is a Radon measure and  $\mu$  is a singleton-finite signed topological measure. If  $|\mu(K)| \leq n(K)$  for any  $K \in \mathcal{K}(X)$  then  $|\mu|$  is a Radon measure.*

*Proof.* For any finite disjoint collection of compact sets  $\bigsqcup_{i=1}^p K_i \subseteq U$  where  $U \in \mathcal{O}(X)$  we have  $\sum_{i=1}^p |\mu(K_i)| \leq \sum_{i=1}^p n(K_i) = n(\bigsqcup_{i=1}^p K_i) \leq n(U)$ , so by Definition 2.16  $|\mu|(U) \leq n(U)$ . From  $|\mu| \leq n$  on  $\mathcal{O}(X)$  we have  $|\mu| \leq n$ . By Theorem 4.1 there exists a singleton-finite deficient topological measure  $\lambda$  such that  $|\mu| + \lambda = n$ . By Theorem 4.3 write unique decompositions  $|\mu| = m + \mu', \lambda = l + \lambda', n = n + 0$ , where  $m, l$  are Radon measures, and  $\mu', \lambda'$  are proper singleton-finite deficient topological measures. Then  $|\mu| + \lambda = (m + l) + (\mu' + \lambda') = n + 0$ . Thus,  $\mu' + \lambda' = 0$ , i.e.  $\mu' = \lambda' = 0$ , and  $|\mu| = m$  is a Radon measure.  $\square$

**Example 5.9.** We will give an example of a proper signed topological measure which is not a (proper) signed Radon measure. Consider a signed topological measure  $\mu = \nu_1 - \nu_3$  from [14, Example 25]. It is easy to represent  $X$  as a finite union of open solid sets each of which contains at most one point from  $P$ . By Theorem 4.4,  $\nu_1$  and  $\nu_3$  are proper topological measures. By Remark 5.2  $\mu$  is a proper signed topological measure. Since  $\mu \neq 0$ , by Remark 3.4  $\mu$  is not a proper signed Radon measure.

**Lemma 5.10.** *Suppose  $\nu$  is a proper singleton-finite signed deficient topological measure. Then  $\nu = 0$  iff*

$$\nu(K \cup C) = \nu(K) + \nu(C) - \nu(K \cap C) \text{ for any compact sets } K, C. \quad (5.1)$$

*Proof.* If  $\nu = 0$  then (5.1) obviously holds. Now assume (5.1). Then it follows that for finitely many compact sets  $K_1, \dots, K_n$  we have

$$\nu\left(\bigcup_{i=1}^n K_i\right) = \sum_{i=1}^n \nu(K_i) - \nu(K_1 \cap K_2) - \nu((K_1 \cup K_2) \cap K_3) - \dots - \nu\left(\left(\bigcup_{i=1}^{n-1} K_i\right) \cap K_n\right).$$

Then  $|\nu(\bigcup_{i=1}^n K_i)| \leq 2 \sum_{i=1}^n |\nu|(K_i)$ . By Remark 3.4 and Theorem 4.4, given a compact set  $K$  and  $\epsilon > 0$ , there are finitely many compact sets  $K_1, \dots, K_n$  such that  $K = \bigcup_{i=1}^n K_i$  and  $\sum_{i=1}^n |\nu|(K_i) < \epsilon$ . Then  $|\nu(K)| = |\nu(\bigcup_{i=1}^n K_i)| \leq 2 \sum_{i=1}^n |\nu|(K_i) \leq 2\epsilon$ . Thus,  $\nu(K) = 0$  for any compact set  $K$ , so  $\nu = 0$ .  $\square$

**Theorem 5.11.** *Suppose a signed singleton-finite deficient topological measure  $\mu$  is the difference of two deficient topological measures, one singleton-finite, the other finite. (In particular,  $\mu$  is a signed topological measure with  $\|\mu\| < \infty$ .) Then  $\mu$  is a signed Radon measure iff equality (5.1) holds for  $\mu$ .*

*Proof.* By Theorem 5.3 there is a decomposition  $\mu = m + \mu'$ , where  $m$  is a signed Radon measure and  $\mu'$  is a proper singleton-finite signed deficient topological measure. Since (5.1) is satisfied for signed measures, we see that (5.1) holds for  $\mu$  iff (5.1) holds for  $\mu'$ , which by Lemma 5.10 happens iff  $\mu' = 0$ , i.e. iff  $\mu = m$  is a signed Radon measure.  $\square$

*Remark 5.12.* If  $X$  is compact and  $\mu$  is a signed topological measure with finite norm, then  $\mu$  is a regular Borel signed measure on  $X$  iff  $\mu(U \cup V) = \mu(U) + \mu(V) - \mu(U \cap V)$  for  $U, V \in \mathcal{O}(X)$ . See [11, Theorem 1].

*Remark 5.13.* When  $X$  is compact and  $\nu$  is a finite topological measure, Theorem 4.3 was first proved by D. Grubb and T. LaBerge, see [12, Theorem 2.1]. When  $X$  is compact and  $\nu$  is a finite deficient topological measure Theorem 4.3 was first proved by M. Svistula, see [17, Theorem 1]. Part (II) of Theorem 4.4, Theorem 4.5, and Theorem 4.7 generalize [17, Theorems 2, 3, and Proposition 7], which in turn generalize corresponding results for topological measures in [16]. Lemma 4.10, Lemma 4.12, and Corollary 4.8 generalize corresponding results from [16, pp. 675, 677]. Part (iii) of Theorem 5.3, Theorem 5.4, Corollary 5.6, Corollary 5.7 generalize [17, Theorem 6, Proposition 9, Corollaries 1, 2, 3]. Lemma 5.10, and Theorem 5.11 generalize [11, Theorem 1] and [17, Propositions 10, 11]. In all previous results the space was compact and all topological measures and deficient topological measures were finite. In our new results the space is locally compact and we no longer require set functions to be finite.

*Remark 5.14.* Topological measures and deficient topological measures correspond to certain nonlinear functionals, see [5]. Thus, in a manner similar to one in [16] and [17], the results involving proper deficient topological measures can be transferred to corresponding nonlinear functionals.

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