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QUALITATIVE ANALYSIS OF ψ–CAPUTO FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. In this paper, we study the existence, uniqueness, and the Mittag-Leffler–Ulam stability of solutions for some classes of hyperbolic fractional order differential equations involving the ψ-Caputo fractional partial derivative. Our main results rely on fixed point theory and the technique of measures of noncompactness. A new generalization of Gronwall's inequality is used to obtain the Mittag-Leffler–Ulam–Hyers–Rassias stability of the proposed problem. Examples illustrating our results are also presented.

1. Introduction

The theory of fractional differential equations has been widely applied in various fields of science and engineering; see [23, 32, 33, 39, 40]. For more information about the fundamental concepts of fractional calculus and fractional differential equations, we suggest the readers consult these books [4–6, 25, 31, 32, 49, 50]. Moreover, some researchers considered new definitions of fractional differential operators; for more details see [11, 36]. Additionally, some interesting details about the ψ-Riemann–Liouville fractional partial integral and the ψ-Hilfer fractional partial derivative can be found in the excellent paper of Sousa *et al.* [37]. In the same context, there are many interesting results for qualitative analysis of fractional ordinary and partial differential equations involving different types of fractional derivatives; for more details see the papers [1–3, 7, 8, 10, 19, 24, 26, 28, 35, 38, 42–48] and the references cited therein.

To our knowledge, hyperbolic fractional partial differential equations in Banach spaces involving the ψ-Caputo fractional partial derivative has not been extensively studied, but partial motivation for this work is the recent paper by the

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authors [14]. Thus, the objective of the present work is to obtain results on existence, uniqueness, and Mittag-Leffler–Ulam–Hyers–Rassias stability of solutions for a ψ-Caputo fractional partial fractional differential equation of the type,

$$\begin{cases} \left({}^{c}\mathbb{D}_{\theta}^{\beta;\psi} x\right)(\tau,\mathrm{u}) = \mathbb{F}(\tau,\mathrm{u},x(\tau,\mathrm{u})), \quad (\tau,\mathrm{u}) \in \tilde{\mathrm{I}} := [a,b]\times[a,c] \\ x(\tau,a) = \eta(\tau), \quad \tau\in[a,b], \\ x(a,\mathrm{u}) = \nu(\mathrm{u}), \quad \mathrm{u}\in[a,c], \\ x(a,a) = \eta(a) = \nu(a), \end{cases} \tag{1.1}$$

where a, b and c are positive constants, $\mathbb{F} : \tilde{\mathrm{I}} \times \mathbb{X} \longrightarrow \mathbb{X}$ is a given function satisfying some assumptions that will be specified later, $\mathbb{X}$ is a real Banach space with norm $\|\cdot\|$, and $\eta : [a,b] \longrightarrow \mathbb{X}$, $\nu : [a,c] \longrightarrow \mathbb{X}$ are given absolutely continuous functions. ${}^{c}\mathbb{D}_{\theta}^{\beta;\psi}$ is the ψ-Caputo fractional derivative of order $\beta = (\beta_1,\beta_2) \in (0,1]\times(0,1]$ and $\theta = (a,a)$.

This paper is designed as follows. Section 2 contains some preliminary concepts related to our problem. In Section 3, we state and prove our main results by using some fixed point theorems, together with the concept of measure of noncompactness, as well as a Gronwall inequality, which plays a vital role in the Mittag-Leffler–Ulam–Hyers–Rassias stability results for the problem at hand. Lastly, in Section 4, we provide two examples to illustrate our main results.

2. Preliminaries and Background Materials

In this section, we present some basic notations, definitions, and preliminary results, which will be used throughout this paper.

Let $\tilde{\mathrm{I}} := [a,b]\times[a,c]$, and we consider $C(\tilde{\mathrm{I}},\mathbb{X})$ the Banach space of all continuous functions x from $\tilde{\mathrm{I}}$ into $\mathbb{X}$ with the norm

$$\|x\|_\infty = \sup_{(\tau,\mathrm{u})\in\tilde{\mathrm{I}}} \big\|x(\tau,\mathrm{u})\big\|.$$

A measurable function $x : \tilde{\mathrm{I}} \to \mathbb{X}$ is Bochner integrable if and only if $\|x\|$ is Lebesgue integrable.

By $L^1(\tilde{\mathrm{I}},\mathbb{X})$ we denote the space of Bochner-integrable functions $x : \tilde{\mathrm{I}} \to \mathbb{X}$, with the norm

$$\|x\|_1 = \int_a^b \int_a^c \big\|x(\tau,\mathrm{u})\big\| \mathrm{d}\tau \mathrm{d}\mathrm{u}.$$

First of all, we present some facts from the theory of fractional calculus.

Definition 2.1 ([37]). Let $\theta = (a,a)$ and $\beta = (\beta_1,\beta_2)$ where $\beta_1,\beta_2 > 0$. Also let $\psi(\cdot)$ be an increasing positive monotone function on each of $(a,b]$ and $(a,c]$ and having continuous derivative $\psi'(\cdot)$ on each of $(a,b]$ and $(a,c]$. The ψ-Riemann–Liouville partial integral of a function $x \in L^1(\tilde{\mathrm{I}},\mathbb{R})$, is defined as

$$\left(\mathbb{I}_{\theta}^{\beta;\psi} x\right)(\tau,\mathrm{u}) = \int_a^{\tau}\int_a^{\mathrm{u}} \frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)} x(s,t)\mathrm{d}t\mathrm{d}s. \tag{2.1}$$

Definition 2.2 ([37]). Let $\theta = (a, a)$ and $\beta = (\beta_1, \beta_2)$ where $0 < \beta_1, \beta_2 \leq 1$. Also let $x \in C^1(\tilde{\mathrm{I}}, \mathbb{R})$, and let ψ belong to both $C^1([a, b], \mathbb{R})$ and $C^1([a, c], \mathbb{R})$ such that ψ is increasing in both cases, and $\psi'(\tau) \neq 0, \psi'(\mathrm{u}) \neq 0$, for all $(\tau, \mathrm{u}) \in \tilde{\mathrm{I}}$. The ψ-Caputo fractional partial derivative of functions of two variables of order β is given by

$$\left({}^c\mathbb{D}_\theta^{\beta;\psi} x\right)(\tau, \mathrm{u}) = \left(\mathbb{I}_\theta^{1-\beta;\psi} x\right)\left(\frac{1}{\psi'(\tau)\psi'(\mathrm{u})}\frac{\partial^2}{\partial\tau\partial\mathrm{u}}\right) x(\tau, \mathrm{u}).$$

Lemma 2.3 ([37]). *Let $\sigma_1, \sigma_2 \in (-1, +\infty)$ and $\beta = (\beta_1, \beta_2) \in (0, \infty) \times (0, \infty)$. Then*

$$\mathbb{I}_\theta^{\beta;\psi}\left((\psi(\tau) - \psi(a))^{\sigma_1}(\psi(\mathrm{u}) - \psi(a))^{\sigma_2}\right) = \frac{\Gamma(1+\sigma_1)\Gamma(1+\sigma_2)}{\Gamma(\beta_1+\sigma_1+1)\Gamma(\beta_2+\sigma_2+1)} \times (\psi(\tau) - \psi(a))^{\beta_1+\sigma_1}(\psi(\mathrm{u}) - \psi(a))^{\beta_2+\sigma_2}.$$

Remark 2.4. Note that for an abstract function $x : \tilde{\mathrm{I}} \longrightarrow \mathbb{X}$, the integrals which appear in the previous definitions are taken in Bochner's sense, (see, for instance, [34]).

In the sequel, we will make use of the following generalizations of Gronwall's lemmas.

Lemma 2.5 ([41]). *We assume that:*

(1) $v, w \in C(\tilde{\mathrm{I}}, \mathbb{R}_+)$
(2) *ψ belong to both $C^1([a, b], \mathbb{R})$ and $C^1([a, c], \mathbb{R})$ such that ψ is increasing in both cases, and $\psi'(\tau) \neq 0, \psi'(\mathrm{u}) \neq 0$, for all $(\tau, \mathrm{u}) \in \tilde{\mathrm{I}}$.*

If there are constants $c > 0$ and $0 < \beta_1, \beta_2 < 1$ such that

$$v(\tau, \mathrm{u}) \leq w(\tau, \mathrm{u}) + c\int_a^\tau\int_a^\mathrm{u} \psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}v(s, t)\mathrm{d}t\mathrm{d}s,$$

then, we have

$$v(\tau, \mathrm{u}) \leq w(\tau, \mathrm{u})\mathbb{E}_{(\beta_1,\beta_2)}\left(c\Gamma(\beta_1)\Gamma(\beta_2)(\psi(\tau) - \psi(a))^{\beta_1}(\psi(\mathrm{u}) - \psi(a))^{\beta_2}\right), \tag{2.2}$$

where $\mathbb{E}_{(\beta_1,\beta_2)}(\cdot)$ is the Mittag-Leffler function [21], defined by

$$\mathbb{E}_{(\beta_1,\beta_2)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta_1 k + 1)\Gamma(\beta_2 k + 1)}, \quad (z \in \mathbb{R},\ \beta_1, \beta_2 > 0).$$

Next, we define the Kuratowski measure of noncompactness and give some of its important properties.

Definition 2.6 ([16]). Let $\mathbb{X}$ be a Banach space and $\mathfrak{M}_\mathbb{X}$ be the bounded subsets of $\mathbb{X}$. The mapping $\kappa\colon \mathfrak{M}_\mathbb{X} \longrightarrow [0, \infty)$ for Kuratowski measure of noncompactness is defined as,

$$\kappa(B) = \inf\left\{\varepsilon > 0 : B \text{ can be covered by finitely many sets with diameter} \leq \varepsilon\right\}.$$

Lemma 2.7 ([16]). *The Kuratowski measure of noncompactness satisfies some properties.*

(1) $\kappa(A) \leq \kappa(B)$ *if* $A \subset B$;

(2) $\kappa(A) = \kappa(\overline{A}) = \kappa(conv\, A)$, *where conv A means the convex hull of A;*

(3) $\kappa(A) = 0$ *if and only if* $\overline{A}$ *is compact, where* $\overline{A}$ *means the closure hull of* A;

(4) $\kappa(\lambda A) = |\lambda|\kappa(A)$, *where* $\lambda \in \mathbb{R}$;

(5) $\kappa(A \cup B) = \max\{\kappa(A), \kappa(B)\}$;

(6) $\kappa(A + B) \leq \kappa(A) + \kappa(B)$, *where* $A + B = \{w \mid w = a + b, a \in A, b \in B\}$;

(7) $\kappa(A + x) = \kappa(A)$, *for any* $x \in \mathbb{X}$.

(8) *If the map* $\mathbb{T} : \mathrm{dom}(\mathbb{T}) \subset \mathbb{X} \to \mathbb{X}$ *is Lipschitz continuous with constant* k, *then* $\kappa(\mathbb{T}(B)) \leq \mathrm{k}\kappa(B)$ *for any bounded subset* $B \subset \mathrm{dom}(\mathbb{T})$.

The following lemmas are needed in our argument.

Lemma 2.8 ([17]). *Let* $\mathbb{X}$ *be a Banach space and let* $B \subset \mathbb{X}$ *be bounded. Then for each* ε, *there is a sequence* $\{x_n\}_{n=1}^{\infty} \subset B$, *such that*

$$\kappa(B) \leq 2\kappa(\{x_n\}_{n=1}^{\infty}) + \varepsilon.$$

We call $B \subset L^1(\tilde{\mathrm{I}}, \mathbb{X})$ uniformly integrable if there exists $\varphi \in L^1(\tilde{\mathrm{I}}, \mathbb{R}^+)$ such that

$$\|x(\tau, \mathrm{u})\| \leq \varphi(\tau, \mathrm{u}), \text{ for all } x \in B \text{ and a.e. } (\tau, \mathrm{u}) \in \tilde{\mathrm{I}}.$$

Lemma 2.9 ([22]). *If* $\{x_k\}_{k=1}^{\infty} \subset L^1(\tilde{\mathrm{I}}, \mathbb{X})$ *is uniformly integrable, then the function* $\kappa(\{x_k\}_{k=1}^{\infty})$ *is measurable and for each* $(\tau, \mathrm{u}) \in \tilde{\mathrm{I}}$,

$$\kappa\Big(\Big\{\int_a^{\tau}\int_a^{\mathrm{u}} x_k(s,t)\mathrm{dtds}\Big\}_{k=1}^{\infty}\Big) \leq 2\int_a^{\tau}\int_a^{\mathrm{u}} \kappa(\{x_k(s,t)\}_{k=1}^{\infty})\mathrm{dtds}.$$

Lemma 2.10 ([20]). *Let* $\mathcal{V} \subset C(\tilde{\mathrm{I}}, \mathbb{X})$ *be a bounded and equi-continuous subset. Then the function* $(\tau, \mathrm{u}) \to \kappa(\mathcal{V}(\tau, \mathrm{u}))$ *is continuous on* $\tilde{\mathrm{I}}$, *and*

$$\kappa_C(\mathcal{V}) = \max_{(\tau,\mathrm{u})\in\tilde{\mathrm{I}}} \kappa(\mathcal{V}(\tau, \mathrm{u})),$$

and

$$\kappa\left(\left\{\int_a^{\tau}\int_a^{\mathrm{u}} x(s,t)\mathrm{dtds}\right\}\right) \leq \int_a^{\tau}\int_a^{\mathrm{u}} \kappa(\mathcal{V}(s,t))\mathrm{dtds},$$

where $\mathcal{V}(s,t) = \{x(s,t) : x \in \mathcal{V}\}, (s,t) \in \tilde{\mathrm{I}}$, *and* κ_C *is the Kuratowski measure of noncompactness defined on the bounded sets of* $C(\tilde{\mathrm{I}}, \mathbb{X})$.

The following fixed point theorems will be very useful in proving our main results.

Theorem 2.11. (*Mönch's fixed point theorem [30]*). *Let* D *be a bounded, closed and convex subset of a Banach space* $\mathbb{X}$ *such that* $0 \in D$, *and let* $\mathbb{T}$ *be a continuous mapping of* D *into itself. If the implication*

$$\mathcal{V} = \overline{conv}\mathbb{T}(\mathcal{V}), \text{ or } \mathcal{V} = \mathbb{T}(\mathcal{V}) \cup \{0\} \Rightarrow \kappa(\mathcal{V}) = 0, \tag{2.3}$$

holds for every subset $\mathcal{V} \subset D$, *then* $\mathbb{T}$ *has a fixed point.*

On the other hand, in 1969, the concepts of Meir–Keeler contraction mapping were introduced by Meir and Keeler.

Definition 2.12 ([29]). Let $(\mathbb{E}, d)$ be a metric space. Then a mapping $\mathbb{T}$ on $\mathbb{E}$ is said to be a Meir–Keeler contraction (MKC, for short) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(\mathbb{T}x, \mathbb{T}y) < \varepsilon, \quad \forall x, y \in \mathbb{E}.$$

In [9], the authors defined the notion of Meir–Keeler condensing operators on a Banach space and gave some fixed point results.

Definition 2.13 ([9]). Let C be a nonempty subset of a Banach space $\mathbb{X}$ and μ an arbitrary measure of noncompactness on $\mathbb{X}$. We say that an operator $\mathbb{T} : C \to C$ is a Meir–Keeler condensing operator if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq \mu(\Omega) < \varepsilon + \delta \Rightarrow \mu(\mathbb{T}\Omega) < \varepsilon,$$

for any bounded subset Ω of C.

The following fixed point theorem with respect to the Meir–Keeler condensing operator, which introduced by Aghajani *et al.* [9], plays a key role in the proof of our main results.

Theorem 2.14 ([9]). *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space $\mathbb{X}$ Also, let μ be an arbitrary measure of noncompactness on $\mathbb{X}$. If $\mathbb{T} : \Omega \to \Omega$ is a continuous and Meir–Keeler condensing operator, then $\mathbb{T}$ has at least one fixed point, and the set of all fixed points of $\mathbb{T}$ in Ω is compact.*

Now, we consider the $\mathbb{E}_{(\beta_1,\beta_2)}$–Ulam–Hyers stability for problem (1.1). Let $\varepsilon > 0, \mathbb{L} \geq 0$ and $\Phi : \tilde{\mathrm{I}} \to \mathbb{R}^+$, be a continuous function. We consider the following inequalities:

$$\left\|\left({}^c\mathbb{D}_\theta^{\beta;\psi} y\right)(\tau, \mathrm{u}) - \mathbb{F}(\tau, \mathrm{u}, y(\tau, \mathrm{u}))\right\| \leq \varepsilon, \quad (\tau, \mathrm{u}) \in \tilde{\mathrm{I}}; \tag{2.4}$$

$$\left\|\left({}^c\mathbb{D}_\theta^{\beta;\psi} y\right)(\tau, \mathrm{u}) - \mathbb{F}(\tau, \mathrm{u}, y(\tau, \mathrm{u}))\right\| \leq \Phi(\tau, \mathrm{u}), \quad (\tau, \mathrm{u}) \in \tilde{\mathrm{I}}; \tag{2.5}$$

$$\left\|\left({}^c\mathbb{D}_\theta^{\beta;\psi} y\right)(\tau, \mathrm{u}) - \mathbb{F}(\tau, \mathrm{u}, y(\tau, \mathrm{u}))\right\| \leq \varepsilon\Phi(\tau, \mathrm{u}), \quad (\tau, \mathrm{u}) \in \tilde{\mathrm{I}}. \tag{2.6}$$

Definition 2.15 ([47, 48]). Equation (1.1) is $\mathbb{E}_{(\beta_1,\beta_2)}$–Ulam–Hyers stable if there exists a real number $c > 0$ such that, for each $\varepsilon > 0$ and for each solution $y \in C(\tilde{\mathrm{I}}, \mathbb{X})$ of inequalitiy (2.4), there exists a solution $x \in C(\tilde{\mathrm{I}}, \mathbb{X})$ of (1.1) with

$$\|y(\tau, \mathrm{u}) - x(\tau, \mathrm{u})\| \leq c\varepsilon\mathbb{E}_{(\beta_1,\beta_2)}\big(\mathbb{L}(\psi(\tau) - \psi(a))^{\beta_1}(\psi(\mathrm{u}) - \psi(a))^{\beta_2}\big).$$

Definition 2.16 ([47, 48]). Equation (1.1) is generalized $\mathbb{E}_{(\beta_1,\beta_2)}$–Ulam–Hyers stable if there exists $\omega : C(\mathbb{R}_+, \mathbb{R}_+)$ with $\omega(0) = 0$ such that, for each $\varepsilon > 0$ and for each solution $y \in C(\tilde{\mathrm{I}}, \mathbb{R})$ of inequalitiy (2.4), there exists a solution $x \in C(\tilde{\mathrm{I}}, \mathbb{X})$ of (1.1) with

$$\|y(\tau, \mathrm{u}) - x(\tau, \mathrm{u})\| \leq \omega(\varepsilon)\mathbb{E}_{(\beta_1,\beta_2)}\big(\mathbb{L}(\psi(\tau) - \psi(a))^{\beta_1}(\psi(\mathrm{u}) - \psi(a))^{\beta_2}\big), \quad (\tau, \mathrm{u}) \in \tilde{\mathrm{I}}.$$

Definition 2.17 ([47,48])**.** Equation (1.1) is $\mathbb{E}_{(\beta_1,\beta_2)}$–Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_\Phi > 0$ such that, for each $\varepsilon > 0$ and for each solution $y \in C(\tilde{\mathrm{I}}, \mathbb{X})$ of inequality (2.6), there exists a solution $x \in C(\tilde{\mathrm{I}}, \mathbb{X})$ of (1.1) with

$$\|y(\tau,\mathrm{u})-x(\tau,\mathrm{u})\| \leq c_\Phi \varepsilon \Phi(\tau,\mathrm{u}) \mathbb{E}_{(\beta_1,\beta_2)}\big(\mathbb{L}(\psi(\tau)-\psi(a))^{\beta_1}(\psi(\mathrm{u})-\psi(a))^{\beta_2}\big), (\tau,\mathrm{u}) \in \tilde{\mathrm{I}}.$$

Definition 2.18 ([47,48])**.** Equation (1.1) is generalized $\mathbb{E}_{(\beta_1,\beta_2)}$–Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_\Phi > 0$ such that, for each solution $y \in C(\tilde{\mathrm{I}}, \mathbb{X})$ of inequality (2.5), there exists a solution $x \in C(\tilde{\mathrm{I}}, \mathbb{X})$ of (1.1) with

$$\|y(\tau,\mathrm{u})-x(\tau,\mathrm{u})\| \leq c_\Phi \Phi(\tau,\mathrm{u}) \mathbb{E}_{(\beta_1,\beta_2)}\big(\mathbb{L}(\psi(\tau)-\psi(a))^{\beta_1}(\psi(\mathrm{u})-\psi(a))^{\beta_2}\big), (\tau,\mathrm{u}) \in \tilde{\mathrm{I}}.$$

Remark 2.19 ([47,48]). It is clear that

(i) Definition 2.15 $\Rightarrow$ Definition 2.16,
(ii) Definition 2.17 $\Rightarrow$ Definition 2.18,
(iii) Definition 2.17 for $\Phi(\cdot,\cdot) = 1 \Rightarrow$ Definition 2.15.

Remark 2.20 ([47,48]). A function $y \in C(\tilde{\mathrm{I}}, \mathbb{X})$ is a solution of inequality (2.6) if and only if there exists a function $g \in C(\tilde{\mathrm{I}}, \mathbb{X})$ (which depends on solution y) such that

(i) $\|g(\tau,\mathrm{u})\| \leq \varepsilon \Phi(\tau,\mathrm{u}), \quad (\tau,\mathrm{u}) \in \tilde{\mathrm{I}}$.
(ii) $({}^c\mathbb{D}_\theta^{\beta;\psi} y)(\tau,\mathrm{u}) = \mathbb{F}(\tau,\mathrm{u},y(\tau,\mathrm{u})) + g(\tau,\mathrm{u}), \quad (\tau,\mathrm{u}) \in \tilde{\mathrm{I}}$.

3. Main results

Let us start by defining what we mean by a solution of problem (1.1).

Definition 3.1 ([43])**.** A function $x \in C(\tilde{\mathrm{I}}, \mathbb{X})$ is said to be a solution of (1.1) if x satisfies the conditions $x(\tau,a) = \eta(\tau)$, $x(a,\mathrm{u}) = \nu(\mathrm{u})$, with $x(a,a) = \eta(a) = \nu(a)$ and the equation $\big({}^c\mathbb{D}_\theta^{\beta;\psi} x\big)(\tau,\mathrm{u}) = \mathbb{F}(\tau,\mathrm{u},x(\tau,\mathrm{u}))$ on $\tilde{\mathrm{I}}$.

To prove our main result we need the following lemma:

Lemma 3.2 ([43])**.** *A function $x \in C(\tilde{\mathrm{I}}, \mathbb{X})$ is a solution of problem (1.1) if and only if x satisfies the following integral equation*

$$\begin{aligned} x(\tau,\mathrm{u}) =& \zeta(\tau,\mathrm{u}) + \int_a^\tau \int_a^{\mathrm{u}} \frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)} \\ & \times \mathbb{F}(s,t,x(s,t))\,\mathrm{d}t\mathrm{d}s, \quad (\tau,\mathrm{u}) \in \tilde{\mathrm{I}}, \end{aligned} \tag{3.1}$$

where

$$\zeta(\tau,\mathrm{u}) = \eta(\tau) + \nu(\mathrm{u}) - \eta(a).$$

We introduce the following conditions:

(H1) The function $\mathbb{F} : \tilde{\mathrm{I}} \times \mathbb{X} \longrightarrow \mathbb{X}$ is continuous.

(H2) There exists $\mathbb{L} > 0$ such that

$$\|\mathbb{F}(\tau,\mathrm{u},x) - \mathbb{F}(\tau,\mathrm{u},y)\| \leq \mathbb{L}\|x-y\|, \quad (\tau,\mathrm{u}) \in \tilde{\mathrm{I}}, x,y \in \mathbb{X}.$$

(H3) There exists $\lambda_\Phi > 0$ such that for each $(\tau, \mathrm{u}) \in \tilde{\mathrm{I}}$, we have

$$\big(\mathbb{I}_\theta^{\beta;\psi}\Phi\big)(\tau, \mathrm{u}) \leq \lambda_\Phi \Phi(\tau, \mathrm{u}).$$

Set

$$\mathbb{M}_\psi = \frac{(\psi(b) - \psi(a))^{\beta_1}(\psi(\mathrm{c}) - \psi(a))^{\beta_2}}{\Gamma(\beta_1 + 1)\Gamma(\beta_2 + 1)},$$

and

$$\mathbb{F}^* = \sup_{(\tau,\mathrm{u})\in\tilde{\mathrm{I}}} \|\mathbb{F}(\tau, \mathrm{u}, 0)\|.$$

Now, we are ready to present our main results.

The first existence result is based on Weissinger's fixed point theorem.

Theorem 3.3. *Assume that* (H1) *and* (H2) *hold. Then there exists a unique solution of problem* (1.1) *on* $\tilde{\mathrm{I}}$. *Furthermore, if hypothesis* (H3) *holds, then the problem* (3.1) *is* $\mathbb{E}_{(\beta_1,\beta_2)}$ *Ulam–Hyers–Rassias stable and consequently generalized* $\mathbb{E}_{(\beta_1,\beta_2)}$*–Ulam–Hyers–Rassias stable.*

Proof. In view of Lemma 3.2, we transform the integral representation (3.1) of the initial value problem (1.1) into

$$x = \mathbb{T}x, \quad x \in C(\tilde{\mathrm{I}}, \mathbb{X}),$$

where $\mathbb{T} : C(\tilde{\mathrm{I}}, \mathbb{X}) \longrightarrow C(\tilde{\mathrm{I}}, \mathbb{X})$ is defined by

$$\begin{aligned} &\mathbb{T}x(\tau, \mathrm{u}) \\ =&\zeta(\tau, \mathrm{u}) + \int_a^\tau \int_a^{\mathrm{u}} \frac{\psi'(s)\psi'(t)(\psi(\tau) - \psi(s))^{\beta_1-1}(\psi(\mathrm{u}) - \psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)} \\ &\times\mathbb{F}(s, t, x(s, t))\mathrm{d}t\mathrm{d}s. \end{aligned} \tag{3.2}$$

Clearly the operator $\mathbb{T}$ is well-defined. Now, we apply Weissinger's fixed point theorem to prove that $\mathbb{T}$ has a unique fixed point. Indeed, it enough to show that $\mathbb{T}^n$ is a contraction operator for sufficiently large n. By the induction method, for any $x, y \in C(\tilde{\mathrm{I}}, \mathbb{X})$ and $(\tau, \mathrm{u}) \in \tilde{\mathrm{I}}$ we will verify that

$$\begin{aligned} &\|\mathbb{T}^n x - \mathbb{T}^n y\|_\infty \\ &\leq \frac{\mathbb{L}^n(\psi(\tau) - \psi(a))^{n\beta_1}(\psi(\mathrm{u}) - \psi(a))^{n\beta_2}}{\Gamma(n\beta_1 + 1)\Gamma(n\beta_2 + 1)}\|x - y\|_\infty, \quad n \in \mathbb{N}. \end{aligned} \tag{3.3}$$

For $n = 0$, the above inequality is trivially true. We assume that (3.3) is true for $n = k$ and prove it for $n = k + 1$. From the definition of the operator $\mathbb{T}$ and assumption (H2) we can get

$$\begin{aligned}
&\left\|\mathbb{T}^{k+1}x(\tau,\mathrm{u})-\mathbb{T}^{k+1}y(\tau,\mathrm{u})\right\|=\left\|\mathbb{T}(\mathbb{T}^{k}x(\tau,\mathrm{u}))-\mathbb{T}(\mathbb{T}^{k}y(\tau,\mathrm{u}))\right\|\\
&\leq\int_a^\tau\int_a^{\mathrm{u}}\frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)}\\
&\quad\times\left\|\mathbb{F}\big(s,t,\mathbb{T}^{k}x(s,t)\big)-\mathbb{F}\big(s,t,\mathbb{T}^{k}y(s,t)\big)\right\|\mathrm{dtds}\\
&\leq\frac{\mathbb{L}^{k+1}\|x-y\|_\infty}{\Gamma(k\beta_1+1)\Gamma(k\beta_2+1)}\int_a^\tau\int_a^{\mathrm{u}}\frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)}\\
&\quad\times(\psi(s)-\psi(a))^{n\beta_1}(\psi(t)-\psi(a))^{n\beta_2}\mathrm{dtds}\\
&\leq\frac{\mathbb{L}^{k+1}\|x-y\|_\infty}{\Gamma(k\beta_1+1)\Gamma(k\beta_2+1)}\int_a^\tau\frac{\psi'(s)(\psi(\tau)-\psi(s))^{\beta_1-1}}{\Gamma(\beta_1)}(\psi(s)-\psi(a))^{n\beta_1}\mathrm{ds}\\
&\quad\times\int_a^{\mathrm{u}}\frac{\psi'(t)(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_2)}(\psi(t)-\psi(a))^{n\beta_2}\mathrm{dt}.
\end{aligned}$$

Note that,

$$\begin{aligned}
&\int_a^z\frac{\psi'(\mathrm{w})(\psi(z)-\psi(\mathrm{w}))^{v-1}}{\Gamma(v)}\left(\psi(\mathrm{w})-\psi(a)\right)^{kv}\mathrm{dw}\\
&=\frac{(\psi(z)-\psi(a))^{(k+1)v}}{\Gamma(v)}\int_0^1(1-\theta)^{v-1}\theta^{kv}\mathrm{d}\theta\\
&=\frac{(\psi(z)-\psi(a))^{(k+1)v}}{\Gamma(v)}B(v,kv+1)\\
&=\frac{\Gamma(kv+1)\left(\psi(z)-\psi(a)\right)^{(k+1)v}}{\Gamma((k+1)v+1)},
\end{aligned}$$

where we have used the variable substitution $y=\frac{\psi(w)-\psi(a)}{\psi(z)-\psi(a)}$, and the relationship between the the Beta function and the Gamma function.

Using the above arguments, we get

$$\|\mathbb{T}^{k+1}x-\mathbb{T}^{k+1}y\|_\infty\leq\frac{\mathbb{L}^{k+1}(\psi(\tau)-\psi(a))^{(k+1)\beta_1}(\psi(\mathrm{u})-\psi(a))^{(k+1)\beta_2}}{\Gamma((k+1)\beta_1+1)\Gamma((k+1)\beta_2+1)}\|x-y\|_\infty.$$

Therefore, by the method of mathematical induction, we know that the inequality (3.3) is true for any $n\in\mathbb{N}$ and $(\tau,\mathrm{u})\in\tilde{\mathrm{I}}$. Hence, we have

$$\|\mathbb{T}^{n}x-\mathbb{T}^{n}y\|_\infty\leq\frac{\mathbb{L}^{n}(\psi(b)-\psi(a))^{n\beta_1}(\psi(\mathrm{c})-\psi(a))^{n\beta_2}}{\Gamma(n\beta_1+1)\Gamma(n\beta_2+1)}\|x-y\|_\infty,\quad n\in\mathbb{N}.$$

Setting

$$\alpha_n=\frac{\big(\mathbb{L}(\psi(b)-\psi(a))^{\beta_1}(\psi(\mathrm{c})-\psi(a))^{\beta_2}\big)^n}{\Gamma(n\beta_1+1)\Gamma(n\beta_2+1)},$$

we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty}\alpha_n&=\sum_{n=0}^{\infty}\frac{\big(\mathbb{L}(\psi(b)-\psi(a))^{\beta_1}(\psi(\mathrm{c})-\psi(a))^{\beta_2}\big)^n}{\Gamma(n\beta_1+1)\Gamma(n\beta_2+1)}\\
&=\mathbb{E}_{(\beta_1,\beta_2)}\big(\mathbb{L}(\psi(b)-\psi(a))^{\beta_1}(\psi(\mathrm{c})-\psi(a))^{\beta_2}\big).
\end{aligned}$$

Thus, $\mathbb{T}^n$ is a contraction mapping. Therefore by Weissinger's fixed point theorem, $\mathbb{T}$ has a unique fixed point. That is (1.1) has a unique solution.

Now we complete the proof by studying the Mittag-Leffler–Ulam–Hyers–Rassias stability of the proposed problem (1.1).

Let $\varepsilon > 0$, let $y \in C(\tilde{\mathrm{I}}, \mathbb{X})$ be a function which satisfies the inequality (2.6), and let $x \in C(\tilde{\mathrm{I}}, \mathbb{X})$ be the unique solution of the following problem,

$$\begin{cases} \left({}^{c}\mathbb{D}_{\theta}^{\beta;\psi} x\right)(\tau,\mathrm{u}) = \mathbb{F}(\tau,\mathrm{u},x(\tau,\mathrm{u})), \quad (\tau,\mathrm{u}) \in \tilde{\mathrm{I}} := [a,b]\times[a,c], \\ x(\tau,a) = \eta(t), \quad \tau \in [a,b], \\ x(a,\mathrm{u}) = \nu(x), \quad \mathrm{u} \in [a,c], \\ x(a,a) = \eta(a) = \nu(a). \end{cases}$$

By Lemma 3.2, we have

$$\begin{aligned} x(\tau,\mathrm{u}) =& \zeta(\tau,\mathrm{u}) + \int_a^{\tau}\int_a^{\mathrm{u}} \frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)} \\ & \times \mathbb{F}(s,t,x(s,t))\,\mathrm{dtds}. \end{aligned}$$

Since we have assumed that y is a solution of (2.6), hence we have by Remark 2.20,

$$\begin{cases} \left({}^{c}\mathbb{D}_{\theta}^{\beta;\psi} y\right)(\tau,\mathrm{u}) = \mathbb{F}(\tau,\mathrm{u},y(\tau,\mathrm{u})) + g(\tau,\mathrm{u}), \quad (\tau,\mathrm{u}) \in \tilde{\mathrm{I}}, \\ y(\tau,a) = \eta(\tau), \quad \tau \in [a,b], \\ y(a,\mathrm{u}) = \nu(\mathrm{u}), \quad \mathrm{u} \in [a,c], \\ y(a,a) = \eta(a) = \nu(a). \end{cases}$$

Again by Lemma 3.2, we have

$$\begin{aligned} y(\tau,\mathrm{u}) = \zeta(\tau,\mathrm{u}) &+ \int_a^{\tau}\int_a^{\mathrm{u}} \frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)} \\ & \times \big(\mathbb{F}(s,t,y(s,t)) + g(s,t)\big)\mathrm{dtds}. \end{aligned}$$

On the other hand, we have, for each $(\tau,\mathrm{u}) \in \tilde{\mathrm{I}}$

$$\begin{aligned} &\|x(\tau,\mathrm{u}) - y(\tau,\mathrm{u})\| \\ &\leq \int_a^{\tau}\int_a^{\mathrm{u}} \frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)}\|g(s,t)\|\mathrm{dtds} \\ &+ \int_a^{\tau}\int_a^{\mathrm{u}} \frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)} \times \\ &\|\mathbb{F}(s,t,x(s,t)) - \mathbb{F}(s,t,y(s,t))\|\mathrm{dtds} \qquad . \end{aligned}$$

Hence using part (i) of Remark 2.20, (H2) and (H3), we obtain

$$\begin{aligned}
&\|x(\tau,\mathrm{u})-y(\tau,\mathrm{u})\| \\
&\leq \varepsilon\int_a^\tau\int_a^{\mathrm{u}}\frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)}\Phi(s,t)\mathrm{dtds} \\
&+\mathbb{L}\int_a^\tau\int_a^{\mathrm{u}}\frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)}\times \\
&\|x(s,t)-y(s,t)\|\mathrm{dtds} \\
&\leq \varepsilon\lambda_\Phi\Phi(\tau,\mathrm{u})+\mathbb{L}\int_a^\tau\int_a^{\mathrm{u}}\frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)}\times \\
&\|x(s,t)-y(s,t)\|\mathrm{dtds}.
\end{aligned}$$

Applying the Gronwall inequality Eq. (2.2) to the above inequality with

$$v(\tau,\mathrm{u})=\|y(\tau,\mathrm{u})-x(\tau,\mathrm{u})\|,\ \ w(\tau,\mathrm{u})=\varepsilon\lambda_\Phi\Phi(\tau,\mathrm{u}),\ \ c=\frac{\mathbb{L}}{\Gamma(\beta_1)\Gamma(\beta_2)},$$

we have

$$\|y(\tau,\mathrm{u})-x(\tau,\mathrm{u})\|\leq w(\tau,\mathrm{u})\mathbb{E}_{(\beta_1,\beta_2)}\big(\mathbb{L}(\psi(\tau)-\psi(a))^{\beta_1}(\psi(\mathrm{u})-\psi(a))^{\beta_2}\big),$$

which yields that

$$\begin{aligned}
&\|y(\tau,\mathrm{u})-x(\tau,\mathrm{u})\| \\
&\leq \varepsilon\lambda_\Phi\Phi(\tau,\mathrm{u})\mathbb{E}_{(\beta_1,\beta_2)}\big(\mathbb{L}(\psi(\tau)-\psi(a))^{\beta_1}(\psi(\mathrm{u})-\psi(a))^{\beta_2}\big).
\end{aligned}\tag{3.4}$$

Taking for simplicity

$$c_\Phi=\lambda_\Phi,$$

then (3.4) becomes

$$\|y(\tau,\mathrm{u})-x(\tau,\mathrm{u})\|\leq c_\Phi\varepsilon\Phi(\tau,\mathrm{u})\mathbb{E}_{(\beta_1,\beta_2)}\big(\mathbb{L}(\psi(\tau)-\psi(a))^{\beta_1}(\psi(\mathrm{u})-\psi(a))^{\beta_2}\big).$$

Thus, problem (1.1) is $\mathbb{E}_{(\beta_1,\beta_2)}$–Ulam–Hyers–Rassias stable. Further, if we set $\varepsilon=1$, then problem (1.1) is generalized $\mathbb{E}_{(\beta_1,\beta_2)}$–Ulam–Hyers–Rassias stable. This completes the proof. □

The second result is based on Mönch's fixed point theorem.

Theorem 3.4. *Assume that the hypotheses* (H1) *and* (H2) *hold. If*

$$\mathbb{L}\mathbb{M}_\psi<1,\tag{3.5}$$

then problem (1.1) *has at least one solution defined on* $\tilde{\mathrm{I}}$.

Proof. In order to use Mönch's fixed-point theorem, we should define a subset $\mathfrak{B}_r$ of $C(\tilde{\mathrm{I}},\mathbb{X})$ by

$$\mathfrak{B}_r=\big\{w\in C(\tilde{\mathrm{I}},\mathbb{X}):\|w\|_\infty\leq r\big\},$$

with $r>0$, such that

$$r\geq\frac{\|\zeta\|+\mathbb{F}^*\mathbb{M}_\psi}{1-\mathbb{L}\mathbb{M}_\psi}.$$

Notice that $\mathfrak{B}_r$ is a closed, convex and bounded subset of the Banach space $C(\tilde{\mathrm{I}},\mathbb{X})$.

We shall prove that $\mathbb{T}$, satisfies all conditions of Mönch's fixed point theorem.
Step 1: The operator $\mathbb{T}$ maps the set $\mathfrak{B}_r$ into itself. Indeed, For any $x \in \mathfrak{B}_r$ and for each $(\tau, \mathrm{u}) \in \tilde{\mathrm{I}}$, from the definition of the operator $\mathbb{T}$ and assumption (H2), we can get

$$\begin{aligned}\|\mathbb{T}x(\tau,\mathrm{u})\| \leq & \|\zeta(\tau,\mathrm{u})\| + \int_a^\tau \int_a^\mathrm{u} \frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)} \\ & \times \left(\left\|\mathbb{F}(s,t,x(s,t))-\mathbb{F}(s,t,0)\right\|+\left\|\mathbb{F}(s,t,0)\right\|\right)\mathrm{dtds} \\ \leq & \|\zeta\| + (\mathbb{L}r+\mathbb{F}^*) \\ & \times \int_a^\tau \int_a^\mathrm{u} \frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)}\mathrm{dtds} \\ = & \|\zeta\| + \mathbb{M}_\psi(\mathbb{L}r+\mathbb{F}^*).\end{aligned}$$

Thus

$$\|\mathbb{T}x\| \leq r.$$

This proves that $\mathbb{T}$ transforms the ball $\mathfrak{B}_r$ into itself.
Step 2: The operator $\mathbb{T}$ is continuous. Suppose that $\{x_n\}$ is a sequence such that $x_n \to x$ in $\mathfrak{B}_r$ as $n \to \infty$. By (H2), we get

$$\begin{aligned}& \|\mathbb{T}x_n(\tau,\mathrm{u}) - \mathbb{T}x(\tau,\mathrm{u})\| \\ \leq & \mathbb{L}\int_a^\tau \int_a^\mathrm{u} \frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)} \\ & \times \|x_n(s,t)-y_n(s,t)\|\mathrm{dtds}.\end{aligned}$$

Hence

$$\|\mathbb{T}x_n - \mathbb{T}x\| \leq \mathbb{L}\mathbb{M}_\psi\|x_n - x\|. \tag{3.6}$$

Since $x_n \to x$ as $n \to +\infty$, then equation (3.6) implies

$$\|\mathbb{T}x_n - \mathbb{T}x\| \to 0 \ \text{ as } \ n \to +\infty,$$

which implies the continuity of the operator $\mathbb{T}$.
Step 3: $\mathbb{T}(\mathfrak{B}_r)$ is equi-continuous.
Let $(\tau_1, \mathrm{u}_1), (\tau_2, \mathrm{u}_2) \in \tilde{\mathrm{I}}$, with $\tau_1 < \tau_2$ and $\mathrm{u}_1 < \mathrm{u}_2$, and let $x \in \mathfrak{B}_r$. Taking (H2) into consideration we get

$$
\begin{aligned}
&\|\mathbb{T}x(\tau_2,\mathrm{u}_2)-\mathbb{T}x(\tau_1,\mathrm{u}_1)\| \le |\zeta(\tau_2,\mathrm{u}_2)-\zeta(\tau_1,\mathrm{u}_1)|\\
&+\frac{\mathbb{L}r+\mathbb{F}^*}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_a^{\tau_1}\int_a^{\mathrm{u}_1}\psi'(s)\psi'(t)\big[(\psi(\tau_1)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u}_1)-\psi(t))^{\beta_2-1}\\
&\quad-(\psi(\tau_2)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u}_2)-\psi(t))^{\beta_2-1}\big]\mathrm{dtds}\\
&+\frac{\mathbb{L}r+\mathbb{F}^*}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_{\tau_1}^{\tau_2}\int_{\mathrm{u}_1}^{\mathrm{u}_2}\psi'(s)\psi'(t)(\psi(\tau_2)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u}_2)-\psi(t))^{\beta_2-1}\mathrm{dtds}\\
&+\frac{\mathbb{L}r+\mathbb{F}^*}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_{\tau_1}^{\tau_2}\int_{a}^{\mathrm{u}_1}\psi'(s)\psi'(t)(\psi(\tau_2)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u}_2)-\psi(t))^{\beta_2-1}\mathrm{dtds}\\
&+\frac{\mathbb{L}r+\mathbb{F}^*}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_{a}^{\tau_1}\int_{\mathrm{u}_1}^{\mathrm{u}_2}\psi'(s)\psi'(t)(\psi(\tau_2)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u}_2)-\psi(t))^{\beta_2-1}\mathrm{dtds}\\
&\le|\zeta(\tau_2,\mathrm{u}_2)-\zeta(\tau_1,\mathrm{u}_1)|\\
&+\frac{2(\mathbb{L}r+\mathbb{F}^*)(\psi(\mathrm{u}_2)-\psi(\mathrm{u}_1))^{\beta_2}\big[(\psi(\tau_2)-\psi(a))^{\beta_1}-(\psi(\tau_2)-\psi(\tau_1))^{\beta_1}\big]}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}\\
&+(\mathbb{L}r+\mathbb{F}^*)\\
&\times\frac{\big[(\psi(\mathrm{u}_2)-\psi(a))^{\beta_2}(\psi(\tau_2)-\psi(\tau_1))^{\beta_1}-(\psi(\mathrm{u}_1)-\psi(a))^{\beta_2}(\psi(\tau_1)-\psi(a))^{\beta_1}\big]}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}.
\end{aligned}
$$

As $\tau_1\to\tau_2$ and $\mathrm{u}_1\to\mathrm{u}_2$, the right-hand side of the above inequality tends to zero independently of $x\in\mathfrak{B}_r$. Hence, we conclude that $\mathbb{T}(\mathfrak{B}_r)\subseteq C(\tilde{\mathrm{I}},\mathbb{X})$ is bounded and equi-continuous.

Step 4: Our aim in this step is to show that the operator $\mathbb{T}$ satisfies the Mönch condition on $\mathfrak{B}_r$.

Let $\mathcal{V}$ be a subset of $\mathfrak{B}_r$ such that $\mathcal{V}\subset\overline{\mathrm{conv}}(\mathbb{T}(\mathcal{V})\cup\{0\})$. $\mathcal{V}$ is bounded and equi-continuous and therefore the function $(\tau,\mathrm{u})\to\gamma(\tau,\mathrm{u})=\kappa(\mathcal{V}(\tau,\mathrm{u}))$ is continuous on $\tilde{\mathrm{I}}$. From (H2) and the last part of Lemma 2.7, Lemma 2.10 and the properties of the measure κ, we find

$$
\begin{aligned}
&\gamma(\tau,\mathrm{u})\le\kappa(\overline{\mathrm{conv}}(\mathbb{T}(\mathcal{V})(\tau,\mathrm{u})\cup\{0\})\le\kappa(\mathbb{T}(\mathcal{V})(\tau,\mathrm{u}))\\
&\le\kappa\Big\{\int_a^{\tau}\int_a^{\mathrm{u}}\frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)}\mathbb{F}(s,t,x(s,t))\mathrm{dtds}\Big\}\\
&\le\int_a^{\tau}\int_a^{\mathrm{u}}\frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)}\\
&\times\kappa\big\{\mathbb{F}(s,t,x(s,t)),x\in\mathcal{V}\big\}\mathrm{dtds}\\
&\le\mathbb{L}\int_a^{\tau}\int_a^{\mathrm{u}}\frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)}\gamma(s,t)\mathrm{dtds}.
\end{aligned}
$$

Hence, thanks to the Gronwall inequality (Lemma 2.5), we obtain $\gamma(\tau,\mathrm{u})=\kappa(\mathcal{V}(\tau,\mathrm{u}))=0$, for each $(\tau,\mathrm{u})\in\tilde{\mathrm{I}}$, and then $\mathcal{V}(\tau,\mathrm{u})$ is relatively compact in $\mathbb{X}$. In view of the Ascoli-Arzelà theorem, $\mathcal{V}$ is relatively compact in $\mathfrak{B}_r$. By Theorem 2.11, we conclude that $\mathbb{T}$ has a fixed point which is a solution to problem (1.1). This completes the proof. □

Our last result is based on the concept of Meir-Keeler condensing operators.

Theorem 3.5. *Assume that the hypotheses* (H1) *and* (H2) *hold. If*

$$4\mathbb{L}\mathbb{M}_\psi < 1, \tag{3.7}$$

then the IVP (1.1) *has at least one solution defined on* $\tilde{\mathrm{I}}$.

Proof. Consider the operator $\mathbb{T}$ defined in (3.2). We shall show that $\mathbb{T}$ satisfies all the assumptions of Theorem 2.14. We know that $\mathbb{T}:\mathfrak{B}_r\to\mathfrak{B}_r$ is bounded and continuous. It's enough to prove that $\mathbb{T}$ is a Meir–Keeler condensing operator. To do this, suppose $\varepsilon>0$ is given, and we will prove that there exists $\delta>0$ such that

$$\varepsilon \le \kappa_C(B) < \varepsilon+\delta \Rightarrow \kappa_C(\mathbb{T}B)<\varepsilon, \quad \text{for any } B\subset\mathfrak{B}_r.$$

For every bounded subset $B\subset\mathfrak{B}_r$ and $\varepsilon'>0$, using Lemma 2.8 and the properties of κ, there exists a sequence $\{x_n\}_{n=1}^\infty\subset B$ such that

$$\kappa\left(\mathbb{T}(B)(\tau,\mathrm{u})\right)\le$$
$$2\kappa\Big\{\int_a^\tau\int_a^{\mathrm{u}}\frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)}\mathbb{F}\big(s,t,x_n(s,t)\big)\mathrm{d}t\mathrm{d}s\Big\}+\varepsilon'.$$

Next, by Lemma 2.10, (H2) and the last part of Lemma 2.7, we have

$$\begin{aligned}&\kappa\left(\mathbb{T}(B)(\tau,\mathrm{u})\right)\\ \le&4\int_a^\tau\int_a^{\mathrm{u}}\frac{\psi'(s)\psi'(t)(\psi(\tau)-\psi(s))^{\beta_1-1}(\psi(\mathrm{u})-\psi(t))^{\beta_2-1}}{\Gamma(\beta_1)\Gamma(\beta_2)}\\ &\times\kappa\big(\mathbb{F}\big(s,t,\{x_n(s,t)\}_{n=1}^\infty\big)\big)\mathrm{d}t\mathrm{d}s+\varepsilon'\\ \le&4\mathbb{L}\mathbb{M}_\psi\kappa_C(B)+\varepsilon'.\end{aligned}$$

As the last inequality is true, for every $\varepsilon'>0$, we infer

$$\kappa\left(\mathbb{T}(B)(\tau,\mathrm{u})\right)\le 4\mathbb{L}\mathbb{M}_\psi\kappa_C(B).$$

Since $\mathbb{T}(B)\subset\mathfrak{B}_r$ is bounded and equicontinuous, we know from Lemma 2.8 that

$$\kappa_C\left(\mathbb{T}(B)\right)=\max_{t\in\mathrm{J}}\kappa\left(\mathbb{T}(B)(\tau,\mathrm{u})\right).$$

Therefore, we have

$$\kappa_C\left(\mathbb{T}(B)\right)\le 4\mathbb{L}\mathbb{M}_\psi\kappa_C(B).$$

Observe that from the last estimates

$$\kappa_C\left(\mathbb{T}(B)\right)\le 4\mathbb{L}\mathbb{M}_\psi\kappa_C(B)<\varepsilon\Rightarrow\kappa_C(B)<\frac{1}{4\mathbb{L}\mathbb{M}_\psi}\varepsilon.$$

Let us now take

$$\delta=\frac{1-4\mathbb{L}\mathbb{M}_\psi}{4\mathbb{L}\mathbb{M}_\psi}\varepsilon,$$

we get

$$\varepsilon\le\kappa_C(B)<\varepsilon+\delta,$$

which means that $\mathbb{T}:\mathfrak{B}_r\to\mathfrak{B}_r$ is a Meir–Keeler condensing operator. It follows from Theorem 2.14 that the operator $\mathbb{T}$ defined by (3.2) has at least one fixed point $x \in \mathfrak{B}_r$, which is the desired solution of problem (1.1). This completes the proof of Theorem 3.3. □

4. Applications

In this section, we give two examples to illustrate our above results.

Example 4.1. Let

$$\mathbb{X} = \ell^1 = \left\{ x = (x_1, x_2, \dots, x_k, \dots), \quad \sum_{k=1}^{\infty} |x_k| < \infty \right\},$$

be the Banach space with the norm

$$\|x\|_{\ell^1} = \sum_{k=1}^{\infty} |x_k|.$$

Consider the following infinite system of partial hyperbolic fractional differential equations of the form

$$\begin{cases} \left({}^{C}_{H}\mathbb{D}^{\beta}_{\theta} x_k \right)(\tau, \mathrm{u}) = \frac{1}{\tau+\mathrm{u}} \frac{x_k(\tau,\mathrm{u})}{1+|x_k(\tau,\mathrm{u})|}, & (\tau, \mathrm{u}) \in \tilde{\mathrm{I}} := [1, e] \times [1, e], \\ x(\tau, 1) = (\tau, 0, \dots, 0, \dots), & \tau \in [1, e], \\ x(1, \mathrm{u}) = (\mathrm{u}, 0, \dots, 0, \dots), & \mathrm{u} \in [1, e], \end{cases} \tag{4.1}$$

where

$$\beta = (\beta_1, \beta_2) = (0.5, 0.5), \quad a = 1, \quad b = c = e, \quad \psi(\cdot) = \ln(\cdot), \quad \mathbb{X} = \ell^1.$$

Set

$$x = (x_1, x_2, \dots, x_k, \dots), \quad \mathbb{F} = (\mathbb{F}_1, \mathbb{F}_2, \dots, \mathbb{F}_k, \dots).$$

Clearly, the function $\mathbb{F} : \tilde{\mathrm{I}} \times \ell^1 \longrightarrow \ell^1$ is continuous. Moreover, for any $x, y \in \ell^1$ and $(\tau, \mathrm{u}) \in \tilde{\mathrm{I}}$ we have

$$\|\mathbb{F}(\tau, \mathrm{u}, x) - \mathbb{F}(\tau, \mathrm{u}, y)\|_{\ell^1} \leq \frac{1}{2} \|x - y\|_{\ell^1}.$$

Thus, hypothesis (H2) is satisfied with $\mathbb{L} = \frac{1}{2}$. An application of Theorem 3.3 shows that problem (4.1) has a unique solution in $C(\tilde{\mathrm{I}}, \ell^1)$. Moreover, by letting $\Phi(\tau, \mathrm{u}) = \ln \tau \ln \mathrm{u}$, we get

$${}^{H}\mathbb{I}^{\beta}_{\theta} \Phi(\tau, \mathrm{u}) = \frac{(\ln \tau \ln \mathrm{u})^{1.5}}{\Gamma(2.5)\Gamma(2.5)} \leq \frac{16}{9\pi} \ln \tau \ln \mathrm{u} = \lambda_{\Phi} \Phi(t).$$

So condition (H3) is satisfied with $\Phi(\tau, \mathrm{u}) = \ln \tau \ln \mathrm{u}$ and $\lambda_{\Phi} = \frac{16}{9\pi}$. It follows from Theorem 3.3 that problem (4.1) is $\mathbb{E}_{(\beta_1,\beta_2)}$–Ulam–Hyers–Rassias stable and consequently it is generalized $\mathbb{E}_{(\beta_1,\beta_2)}$–Ulam–Hyers–Rassias stable.

Example 4.2. Now let

$$\mathbb{X} = c_0 = \{ x = (x_1, x_2, \dots, x_k, \dots) : x_k \to 0 \ (k \to \infty) \},$$

be the Banach space of real sequences converging to zero, endowed with its usual norm

$$\|x\|_{c_0} = \sup_{k \geq 1} |x_k|.$$

Consider the following infinite system of partial hyperbolic fractional differential equations of the form

$$\begin{cases} \left({}^{c}\mathbb{D}_{\theta}^{\beta;\psi} x_k \right)(\tau, \mathrm{u}) = \frac{1}{9+e^{\tau+\mathrm{u}}} \left(\frac{1}{k^2} + \ln(1 + |x_k(\tau, \mathrm{u})|) \right), \quad (\tau, \mathrm{u}) \in \tilde{\mathrm{I}} := [0,1] \times [0,1], \\ x(\tau, 0) = (0, 0, \dots, 0, \dots), \qquad \tau \in [0,1], \\ x(0, \mathrm{u}) = (0, 0, \dots, 0, \dots), \qquad \mathrm{u} \in [0,1], \end{cases} \tag{4.2}$$

where

$$\beta = (\beta_1, \beta_2) = (0.5, 0.5), \quad a = 0, \quad b = c = 1, \quad \mathbb{X} = c_0.$$

Taking also $\psi(\cdot) = \sigma(\cdot)$ where $\sigma(\cdot)$ is the Sigmoid function [27] which can be expressed in the following form

$$\sigma(z) = \frac{1}{1 + e^{-z}},$$

a convenience of the Sigmoid function is its derivative

$$\sigma'(z) = \sigma(z)(1 - \sigma(z)).$$

Set

$$x = (x_1, x_2, \dots, x_k, \dots), \quad \mathbb{F} = (\mathbb{F}_1, \mathbb{F}_2, \dots, \mathbb{F}_k, \dots).$$

Clearly, the function $\mathbb{F} : \tilde{\mathrm{I}} \times c_0 \longrightarrow c_0$ is continuous. Moreover, for any $x, y \in c_0$ and $(\tau, \mathrm{u}) \in \tilde{\mathrm{I}}$ we have

$$\|\mathbb{F}(\tau, \mathrm{u}, x) - \mathbb{F}(\tau, \mathrm{u}, y)\|_{c_0} \leq \frac{1}{10} \|x - y\|_{c_0}.$$

Therefore, assumption (H2) is satisfied with $\mathbb{L} = \frac{1}{10}$. We shall check that condition (3.5) is satisfied. Indeed

$$\mathbb{L}\mathbb{M}_{\psi} \simeq 0.03.$$

Consequently, all the hypothesis of Theorem 3.4 are satisfied. Hence the problem (4.2) has at least one solution $x \in C(\tilde{\mathrm{I}}, c_0)$. Moreover, the condition (3.7) is satisfied. Then, by Theorem 3.5, problem (4.2) has at least one solution $x \in C(\tilde{\mathrm{I}}, c_0)$.

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