Canad. J. Appl. Math. 3 (2021), no. 1, 80-97

http://www.prior-sci-pub.com

NONLINEAR ELLIPTIC ANISOTROPIC UNILATERAL PROBLEMS WITH VARIABLE EXPONENTS AND L^1 DATA

HICHEM KHELIFI

Abstract. In this paper, we study the anisotropic Sobolev spaces with variable exponents $W_0^{1,\overrightarrow{p}(.)}(\Omega)$, and the existence of entropy solutions for a class of nonlinear elliptic unitaleral problems associated to the equations of the from $\Big\{ -\sum_{i=1}^N D_i a_i(x,u,\nabla u) - \sum_{i=1}^N D_i \phi_i(u) = f \Big\}$, where $-\sum_{i=1}^N D_i a_i(x,u,\nabla u)$ is a Lery-Lions anisotropic operator from $W_0^{1,\overrightarrow{p}(.)}(\Omega)$ into $(W_0^{1,\overrightarrow{p}(.)}(\Omega))'$, the right hand side f belongs to $L^1(\Omega)$ and $\phi_i \in C^0(\mathbb{R},\mathbb{R})$, $\forall i = 1,...,N$.

1. Introduction

Throughout this paper, Ω is assumed to be a bounded domain in $\mathbb{R}^N$ $N \geq 2$ with smooth boundary $\partial\Omega$. We consider the obstacle problem associated with the following elliptic equations

$$\begin{cases} Au - \text{div}\phi(u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$Au = -\sum_{i=1}^N D_i a_i(x,u,\nabla u),$$

$f \in L^1(\Omega)$, $D_i u = \frac{\partial u}{\partial x_i}$. We suppose that $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions such that for almost every x in Ω and for every $(\sigma,\xi) \in \mathbb{R} \times \mathbb{R}^N$ the

Date: Received: 25 July 2020; Accepted: 07 September 2020.
Key words and phrases. Entropy solutions; Anisotropic Sobolev spaces; Unilateral problems; Variable exponents; truncations.
2010 MSC: 35J60, 35J65.

following assumptions are satisfied for all $i = 1, ..., N$

$$\sum_{i=1}^{N} a_i(x,s,\xi).\xi_i \geq \alpha \sum_{i=1}^{N} |\xi_i|^{p_i(x)}, \tag{1.2}$$

$$|a_i(x,s,\xi)| \leq \beta \left[k_i(x) + |\sigma|^{p_i(x)-1} + |\xi_i|^{p_i(x)-1}\right], \tag{1.3}$$

$$\sum_{i=1}^{N}(a_i(x,s,\xi) - a_i(x,s,\eta))(\xi_i - \eta_i) > 0, \quad \forall \xi \neq \eta, \tag{1.4}$$

where $\beta > 0$, $\alpha > 0$, and $k_i \in L^1(\Omega)$, $i = 1, ..., N$, $p_i : \overline{\Omega} \to (1, +\infty)$ are continuous functions. Moreover, we suppose that

$$\phi_i \in C^0(\mathbb{R}, \mathbb{R}), \quad \text{for } \ i = 1, ..., N. \tag{1.5}$$

The objective of our article is to study the anisotropic unilateral nonlinear elliptic problem in anisotropic Sobolev spaces the associated with the nonlinear problem (1.1). We prove the existence of entropy solutions for the following unilateral anisotropic problem.

$$\begin{cases} u \geq \psi \ \text{ a.e. in } \ \Omega, \\ T_k(u) \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \ \ \forall k > 0, \\ \sum\limits_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u) D_i T_k(u-v)dx + \sum\limits_{i=1}^{N} \int_\Omega \phi_i(u) D_i T_k(u-v)dx \\ \qquad \leq \int_\Omega f T_k(u-v)dx, \\ \forall v \in K_\psi \cap L^\infty(\Omega), \end{cases} \tag{1.6}$$

where K_ψ the convex set

$$K_\psi = \left\{ u \in W_0^{1,\overrightarrow{p}(.)}(\Omega), \ \ u \geq \psi \ \text{ a.e. in } \Omega \right\},$$

with ψ is a measurable function with values in $\overline{\mathbb{R}}$ such that

$$\psi^+ \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega). \tag{1.7}$$

and T_k is the usual truncation function. Note that the existence result is proved by assuming only ϕ is continuous function. If we take $\psi = -\infty$, we obtain the existence results of problem (1.6) in the case of equation. The integrals in (1.6) are well defined: Indeed under the condition (1.3), the function $a_i(x,u,\nabla u)$ is belongs to $L^{p_i'(.)}(\Omega)$ and since $D_i T_k(u-v)$ is belongs to $L^{p_i(.)}(\Omega)$, the first integral in the left hand in (1.6) is well defined. For the second integral in the left hand in (1.6), since $\phi_i(u) D_i T_k(u-v) = 0$ on $\{|u| > \|v\|_\infty + k\}$ and the hypothesis (1.5), $\phi_i(u)$ is bounded in $\{|u| > \|v\|_\infty + k\}$, then the second integral is well defined. Moreover since $f \in L^1(\Omega)$ and $T_k(u-v) \in L^\infty(\Omega)$, the integral in the right hand is well defined.

Anisotropic operators with variable exponents are involved in various branches of applied sciences. In some cases, they provide realistic models for the study of natural phenomena in electro-rheological fluids (see references in [14, 15]). Other important application is related to image processing [13].

When the lower-order term does not appear in (1.1) (i.e. $\phi_i(u) = 0$ for $i = 1, ..., N$) and the right-hand side f belongs to $L^1(\Omega)$ the existence of entropy solutions to problem (1.1) are proved in [17] under the hypothesis $p_i(.)$ is Log-Hölder continuous i.e. there exists a positive constant C such that

$$|p_i(x) - p_i(y)| \leq \frac{C}{-\log|x-y|},$$

for every x, y with $|x-y| \leq \frac{1}{2}$. In [16] the authors as showed the entropy solutions for nonlinear elliptic problem (1.1) with $\overrightarrow{p}(.) = \overrightarrow{p}$,$\phi_i(u) = 0$ $\forall i = 1, ..., N$ and the right-hand side f belongs to $L^1(\Omega)$. If $\overrightarrow{p}(.) = \overrightarrow{p}$ and $\phi_i(u) \neq 0$ the problem (1.1), have been treated in [18].

This paper is organized as follows: Section 2 is deoted to preliminaries and the main result. In Section 3 we prove that the operator of approximate problem is coercive and pseudo-monotone. The main existence results is stated and proved in Section 4.

2. Preliminaries and statement of the main result

2.1. **Preliminaries.** In this sub-section, we recall some facts on anisotropic spaces with variable exponents and we give some of their properties. For further details on the Lebesgue-Sobolev spaces with variable exponents, we refer to [1, 3, 4] and references therein. Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, we denote

$$p^+ = \max_{x\in\overline{\Omega}} p(x), \quad p^- = \min_{x\in\overline{\Omega}} p(x) \tag{2.1}$$

and

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) \mid \ p^- > 1\}.$$

Let $p(.) \in C_+(\overline{\Omega})$. We define the space

$$L^{p(\cdot)}(\Omega) = \left\{u : \Omega \to \mathbb{R}^N \text{ measurable} \mid \int_\Omega |u(x)|^{p(x)} dx < \infty\right\}$$

then the expression

$$\|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 \mid \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\},$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $\left(L^{p(\cdot)}(\Omega),\ \|u\|_{p(\cdot)}\right)$ is a separable Banach space. If $0 < \text{meas}(\Omega) < +\infty$ and $p_1, p_2 \in C_+(\overline{\Omega})$ with $p_1 \leq p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous. Moreover, if $1 < p^- < p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. For all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the Hölder type inequality

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)},$$

holds true. We define the variable exponents Sobolev spaces by

$$W^{1,p(.)}(\Omega) = \left\{u \in L^{p(.)}(\Omega) \quad | \, |\nabla u| \in L^{p(.)}(\Omega)\right\},$$

which is a Banach space equipped with the following norm

$$\|u\|_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

Next, we define $W_0^{1,p(.)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$. Finally, we introduce a natural generalization of the variable exponents Sobolev spaces $W_0^{1,p(.)}(\Omega)$ that will enable us to study with sufficient accuracy problem (1.1). Let $\overrightarrow{p}(\cdot) = (p_1(\cdot), ..., p_N(\cdot))$, where $p_i : \overline{\Omega} \to (1, +\infty)$ are continuous functions. We introduce the anisotropic variable exponents Sobolev spaces

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \{u \in L^{p_i(\cdot)}(\Omega) \mid D_i u \in L^{p_i(\cdot)}(\Omega),\ i = 1, ..., N\},$$

with respect to the norm

$$\|v\|_{1,\overrightarrow{p}(.)} = \sum_{i=1}^{N} \left(\|u\|_{L^{p_i(\cdot)}(\Omega)} + \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}\right). \tag{2.2}$$

We introduce the following notation $p_+^+, p_-^- \in \mathbb{R}^+$ as

$$p_+^+ = \max\{p_1^+, ..., p_N^+\}, \quad p_-^- = \min\{p_1^-, ..., p_N^-\}. \tag{2.3}$$

We denote $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = W_0^{1,1}(\Omega) \cap W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ with respect to the norm (2.2). According to [3], $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a reflexive Banach space.

We introduce the function

$$\overline{p}^*(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{if } \overline{p}(x) < N \\ +\infty, & \text{if } \overline{p}(x) \geq N. \end{cases} \tag{2.4}$$

Theorem 2.1 ([3]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(.) = (p_1(.), p_2(.), ..., p_N(.)) \in (C_+(\overline{\Omega}))^N$. Suppose that*

$$p^+(x) < \overline{p}^*(x) \quad \textit{for all} \quad x \in \overline{\Omega}. \tag{2.5}$$

Then

$$\|u\|_{L^{p^+(\cdot)}(\Omega)} \leq C \sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}, \quad \forall\, u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega),$$

where p^+ is defined as in (2.1), $\overline{p}^*$ as in (2.4), and C is a positive constant independent of u. Thus $\sum_{i=1}^N \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}$ is an equivalent norm on $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Proposition 2.2 ([5]). *Suppose that the hypotheses of Theorem 2.1 are satisfied. Then, for all $u \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ we have*

$$\frac{1}{N^{p_-^- - 1}} \|u\|_{1,\overrightarrow{p}(.)}^{p_-^-} - N \leq \sum_{i=1}^{N} \int_{\Omega} |D_i u|^{p_i(x)} dx \leq N + \|u\|_{1,\overrightarrow{p}(.)}^{p_+^+}. \tag{2.6}$$

Moreover, we consider

$$\mathcal{T}^{1,\overrightarrow{p}(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R},\ \text{measurable such that}\ \ T_k(u) \in W^{1,\overrightarrow{p}(\cdot)}(\Omega),\ \forall k > 0\}.$$

Proposition 2.3 ([8,9]). *Let $u \in \mathcal{T}_0^{1,\overrightarrow{p}(.)}(\Omega)$. Then, there exists a unique measurable function $v_i : \Omega \to \mathbb{R}$ such that*

$$D_iT_k(u) = v_i.\chi_{\{|u|<k\}}, \quad \text{for a.e. } x \in \Omega, \forall k > 0, \forall i = 1, ..., N,$$

where χ_A denotes the characteristic function of a measurable set A. The functions v_i are called the weak partial gradients of u and are still denoted D_iu. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v_i coincides with the standard distributional gradient of u, that is, $v_i = D_iu$.

Lemma 2.4 ([10,17]). *Let $g \in L^{p(.)}(\Omega)$ and $g_n \in L^{p(.)}(\Omega)$ with $\|g_n\|_{p(.)} \leq C$. If $g_n(x) \to g(x)$ almost everywhere in Ω, then $g_n \rightharpoonup g$ in $L^{p(.)}(\Omega)$.*

Lemma 2.5 ([11,17]). *Assume that (1.2)-(1.4) hold and let $(u_n)_n$ be a sequence in $W_0^{1,\overrightarrow{p}(.)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,\overrightarrow{p}(.)}(\Omega)$ and*

$$\sum_{i=1}^{N} \int_{\Omega} (a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u))(D_iu_n - D_iu)dx \to 0.$$

Then, $u_n \to u$ in $W_0^{1,\overrightarrow{p}(.)}(\Omega)$ for a subsequence.

2.2. **Statement of main result.** We will extend the notion of entropy solution, see [12], to problem (1.1) as follows:

Definition 2.6. A function $u \in \mathcal{T}_0^{1,\overrightarrow{p}(.)}(\Omega)$ such that $u \geq \psi$ a. e. in Ω is an entropy solution of the problem (1.1), if

$$\sum_{i=1}^{N} \int_{\Omega} [a_i(x, u, \nabla u)D_iT_k(u - \varphi) + \phi_i(u)D_iT_k(u - \varphi)]dx \leq \int_{\Omega} fT_k(u - \varphi)dx,$$

for all $\varphi \in K_\psi(\Omega) \cap L^\infty(\Omega)$.

Our main result is as follows

Theorem 2.7. *Let $f \in L^1(\Omega)$. Assume (1.2)-(1.5) hold. Then there exists at least an entropy solution of problem (1.1).*

3. Approximate solution

We consider the following approximate problems

$$\begin{cases} u_n \in K_\psi, \\ \sum\limits_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla u_n)D_iT_k(u_n - v)dx + \sum\limits_{i=1}^{N} \int_{\Omega} \phi_i^n(u_n)D_iT_k(u_n - v)dx \\ \qquad \leq \int_{\Omega} fT_k(u_n - v)dx, \\ \forall v \in K_\psi \text{ and } \forall k > 0, \end{cases} \tag{3.1}$$

where $f_n = T_n(f)$ and $\phi_i^n(s) = \phi_i(T_n(s))$.

Let us define the operator A_n from $W_0^{1,\vec{p}(.)}(\Omega)$ into its dual $\left(W_0^{1,\vec{p}(.)}(\Omega)\right)'$, by

$$\langle Au, v\rangle = -\sum_{i=1}^{N}\int_\Omega D_i a_i(x,u,\nabla u).D_i v \quad \forall u\in K_\psi,\ \ \forall v\in W_0^{1,\vec{p}(.)}(\Omega).$$

We define $\Phi_n : K_\psi \to \left(W_0^{1,\vec{p}(.)}(\Omega)\right)'$, by

$$\langle \Phi_n u, v\rangle = \sum_{i=1}^{N}\int_\Omega \phi_i(T_n(u))D_i v dx \quad \forall u\in K_\psi,\ \ \forall v\in W_0^{1,\vec{p}(.)}(\Omega).$$

Lemma 3.1. *The operator $B_n = A+\Phi_n$ from $W_0^{1,\vec{p}(.)}(\Omega)$ into $\left(W_0^{1,\vec{p}(.)}(\Omega)\right)'$ is pseudo-monotone and coercive in the following sense; there exists $v_0\in K_\psi$ such that*

$$|\frac{\langle B_n v, v-v_0\rangle|}{\|v\|_{W_0^{1,\vec{p}(.)}(\Omega)}} \to +\infty \quad \text{if } \|v\|_{W_0^{1,\vec{p}(.)}(\Omega)} \to +\infty \quad \text{and } v\in K_\psi.$$

Proof. By Hölder's inequality and (1.5), we have, for all $u,v\in W_0^{1,\vec{p}(.)}(\Omega)$, we have

$$\begin{aligned}
|\langle \Phi_n u, v\rangle| &\le 2\sum_{i=1}^{N}\|\phi(T_n(u))\|_{L^{p_i'(.)}(\Omega)}\|D_i v\|_{L^{p_i(.)}(\Omega)} \qquad (3.2)\\
&\le 2\sum_{i=1}^{N}\left(\int_\Omega |\phi_i(T_n(u))|^{p_i'(x)}dx+1\right)^{\frac{1}{(p_i')^-}}\|D_i v\|_{L^{p_i(.)}(\Omega)}\\
&\le \max_{1\le i\le N}\left[(\sup_{|s|<n}|\phi_i(s)|+1)^{(p_i')^+}(\text{meas}(\Omega)+1)\right]^{\frac{1}{(p_i')^-}}\sum_{i=1}^{N}\|D_i v\|_{L^{p_i(.)}(\Omega)}\\
&\le C(n)\|v\|_{W_0^{1,\vec{p}(.)}(\Omega)}.
\end{aligned}$$

Moreover, let $v_0\in K_\psi$, thanks to Hölder's inequality and (1.3), we have

$$\begin{aligned}
|\langle Av, v_0\rangle| &\le 2\beta\sum_{i=1}^{N}\left\|k_i(x)+|v|^{p_i(x)-1}+|D_i v|^{p_i(x)-1}\right\|_{L^{p_i'(.)}(\Omega)}\|D_i v_0\|_{L^{p_i(.)}(\Omega)}\\
&\le C_n\|v_0\|_{W_0^{1,\vec{p}(.)}(\Omega)}. \qquad (3.3)
\end{aligned}$$

Then by using (3.2) and (3.3) we conclude that $B_n = A+\Phi_n$ is bounded. For the coercivity, by using (1.2), (2.6), and (3.3), we get

$$\begin{aligned}
|\frac{\langle B_n v, v-v_0\rangle|}{\|v\|_{W_0^{1,\vec{p}(.)}(\Omega)}} &= |\frac{\langle B_n v, v\rangle|}{\|v\|_{W_0^{1,\vec{p}(.)}(\Omega)}} - |\frac{\langle B_n v, v_0\rangle|}{\|v\|_{W_0^{1,\vec{p}(.)}(\Omega)}}\\
&\ge \frac{1}{\|v\|_{W_0^{1,\vec{p}(.)}(\Omega)}}\left[\alpha\sum_{i=1}^{N}\int_\Omega |D_i v|^{p_i(x)}dx - C_1\|v_0\|_{W_0^{1,\vec{p}(.)}(\Omega)}\right]\\
&\ge \frac{C_3}{N^{p_-^- -1}}\|v\|^{p_-^- -1}_{W_0^{1,\vec{p}(.)}(\Omega)} - \frac{1}{\|v\|_{W_0^{1,\vec{p}(.)}(\Omega)}}\left[C_3 N + C_1\|v_0\|_{W_0^{1,\vec{p}(.)}(\Omega)}\right].
\end{aligned}$$

Then

$$|\frac{\langle B_n v, v - v_0\rangle|}{\|v\|_{W_0^{1,\overrightarrow{p}(.)}(\Omega)}} \to +\infty \quad \text{as} \quad \|v\|_{W_0^{1,\overrightarrow{p}(.)}(\Omega)} \to +\infty.$$

It remains to prove that the operator B_n is pseudo-monotone. Let $(u_k)_k$ be a sequence in $W_0^{1,\overrightarrow{p}(.)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u, & \text{in} \quad W_0^{1,\overrightarrow{p}(.)}(\Omega), \\ B_n u_k \rightharpoonup \chi, & \text{in} \quad \left(W_0^{1,\overrightarrow{p}(.)}(\Omega)\right)', \\ \limsup\limits_{k\to\infty} \langle B_n u_k, u_k\rangle \leq \langle \chi, u\rangle. \end{cases}$$

We will prove that

$$\chi = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle \quad \text{as} \quad k \to +\infty.$$

Since $(\phi_i^n(u_k))_k$ is bounded in $L^{p'(.)}(\Omega)$ and $\phi_i^n(u_k) \to \phi_i^n(u)$ a.e. in Ω, we have

$$\phi_i^n(u_k) \to \phi_i^n(u) \quad \text{strongly in} \quad L^{p_i'(.)}(\Omega) \quad \text{as} \quad k \to \infty. \tag{3.4}$$

Using (3.4) and similarly to prove of Lemma 3.3 in [5] we can prove that the operator B_n is pseudo-monotone. □

Proposition 3.2. *Under the conditions* (1.2)-(1.5)*, there exists at least one solution of the problem* (3.1)*.*

Proof. In view of Lemma 3.1, there exists at least one solution $u_n \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ of problem (3.1) (see [2], Theorem 8.2, Chapiter 2). □

4. Proof of the main result

4.1. A priori estimates.

Lemma 4.1. *Assume that* (1.2)-(1.5) *hold, and if u_n is a solution of the approximate problem* (3.1)*. Then there exists a constant C, C' such that*

$$\|T_k(u_n)\|_{W_0^{1,\overrightarrow{p}(.)}(\Omega)} \leq C(1 + k + \|\psi^+\|_\infty)^{\frac{1}{p_-^-}}. \tag{4.1}$$

Proof. Let $v = u_n - \lambda T_k(u_n^+ - \psi^+)$ where $\lambda \geq 0$. Since $v \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ and for all λ small enough, we have $v \in K_\psi$, we take v as test function in problem (3.1), we have

$$\sum_{i=1}^{N} \int_\Omega a_i(x, u_n, \nabla u_n) D_i T_k(u_n^+ - \psi^+) dx \leq \int_\Omega f_n T_k(u_n^+ - \psi^+) dx$$
$$+ \sum_{i=1}^{N} \int_\Omega |\phi_i^n(u_n)||D_i T_k(u_n^+ - \psi^+)| dx.$$

Since $D_iT_k(u_n^+ - \psi^+) = 0$ on the set $\{u_n^+ - \psi^+ > k\}$, we get

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} a_i(x, u_n^+, \nabla u_n^+) D_i u_n^+ dx \\
&\quad \leq \int_{\Omega} f_n T_k(u_n^+ - \psi^+) dx \\
&\qquad + \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^n(u_n)||D_i u_n^+| dx \\
&\qquad + \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^n(u_n)||D_i \psi^+| dx \\
&\qquad + \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |a_i(x, u_n^+, \nabla u_n^+)||D_i \psi^+| dx.
\end{aligned}
$$

By (1.2)-(1.3), and the fact that $u_n^+ \leq k + \|\psi^+\|_\infty$, $T_n(u_n^+) \leq u_n^+$, we obtain

$$
\begin{aligned}
&\alpha \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |D_i u_n^+|^{p_i(x)} dx \\
&\quad \leq \int_{\Omega} f_n T_k(u_n^+ - \psi^+) dx + \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^n(T_{k+\|\psi\|_\infty}(u_n))||D_i \psi^+| dx \\
&\qquad + \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^n(T_{k+\|\psi\|_\infty}(u_n))||D_i u_n^+| dx \\
&\qquad + \beta \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} \left[|k(x)| + |u_n^+|^{p_i(x)-1} + |D_i u_n^+|^{p_i(x)-1}\right] |D_i \psi^+| dx.
\end{aligned}
$$

Thanks to Young's inequalities, and theorem 2.1 we obtain

$$
\begin{aligned}
&\alpha \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |D_i u_n^+|^{p_i(x)} dx \\
&\quad \leq \int_{\Omega} f_n T_k(u_n^+ - \psi^+) dx + C_1 \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^n(T_{k+\|\psi\|_\infty}(u_n))|^{p_i'(x)} dx \\
&\quad + \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^n(T_{k+\|\psi\|_\infty}(u_n))||D_i u_n^+| dx + C_2 \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |k(x)|^{p_i'(x)} dx \\
&\quad + \frac{\varepsilon}{2} \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |D_i u_n^+|^{p_i(x)} dx + 3C_3 \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |D_i \psi^+|^{p_i(x)}
\end{aligned}
$$

Using (1.5), $f_n \leq f \in L^1(\Omega)$, and choosing $\varepsilon = \alpha$, we get

$$\sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |D_i u_n^+|^{p_i(x)} dx \leq C_4 k + C_5 \quad \forall k > 0. \tag{4.2}$$

Since $\{x \in \Omega, \;\; u_n^+ \leq k\} \subset \{x \in \Omega, \;\; u_n^+ - \psi^+ \leq k + \|\psi^+\|_\infty\}$, then

$$\begin{aligned}\sum_{i=1}^{N} \int_{\Omega} |D_i T_k(u_n^+)|^{p_i(x)} dx &= \sum_{i=1}^{N} \int_{\{u_n^+ \leq k\}} |D_i u_n^+|^{p_i(x)} dx \\ &\leq \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k + \|\psi^+\|_\infty\}} |D_i u_n^+|^{p_i(x)} dx.\end{aligned}$$

Thus, by (4.2), we obtain

$$\sum_{i=1}^{N} \int_{\Omega} |D_i T_k(u_n^+)|^{p_i(x)} dx \leq C_4(k + \|\psi^+\|_\infty) + C_5 \quad \forall k > 0. \tag{4.3}$$

By (2.6) and (4.3) we obtain

$$\|T_k(u_n^+)\|_{W_0^{1,\vec{p}(.)}(\Omega)} \leq C_6(k + \|\psi^+\|_\infty)^{\frac{1}{p_-^-}} + C_7 \quad \forall k > 0. \tag{4.4}$$

Similarly, taking $v = u_n + T_k(u_n^-)$ as test function in approximate problem (3.1), we obtain

$$\|T_k(u_n^-)\|_{W_0^{1,\vec{p}(.)}(\Omega)} \leq C_8(1 + k)^{\frac{1}{p_-^-}} \quad \forall k > 0. \tag{4.5}$$

Combining (4.5) and (4.5), we get (4.1). □

4.2. **Strong convergence of truncations.**

Lemma 4.2. *If u_n is a solution of approximate problem (3.1). Then there exists a measurable function u and a subsequence of u_n such that*

$$T_k(u_n) \to T_k(u) \quad \textit{strongly in} \quad W_0^{1,\vec{p}(.)}(\Omega) \quad \textit{and a.e. in } \Omega. \tag{4.6}$$

Proof. We will show that $(u_n)_n$ is a Cauchy sequence in measure. Indeed, by Combining the generalized Hölder type inequality, Theorem 2.1 and (4.1), it follows that

$$\begin{aligned} k\text{meas}\{|u_n| > k\} &\leq \int_{\Omega} |T_k(u_n)| dx \\ &\leq 2(\text{meas}(\Omega) + 1)^{\frac{1}{(p_+^+)'}} \|T_k(u)\|_{L^{p^+(.)}(\Omega)} \\ &\leq C_9 \|T_k(u)\|_{W_0^{1,\vec{p}(.)}(\Omega)} \\ &\leq C_{10}(1 + k + \|\psi^+\|_\infty)^{\frac{1}{p_-^-}} \end{aligned}$$

which yields (since $1 - \frac{1}{p_-^-} > 0$)

$$\text{meas}\{|u_n| > k\} \leq \frac{C_{11}(1 + \|\psi^+\|_\infty)^{\frac{1}{p_-^-}}}{k} + \frac{C_{11}}{k^{1 - \frac{1}{p_-^-}}} \to 0 \quad \text{as} \;\; k \to \infty. \tag{4.7}$$

Moreover, we have, for every $\delta > 0$,

$$\text{meas}\{|u_n - u_m| > \delta\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}. \tag{4.8}$$

Let $\varepsilon > 0$, using (4.7) we may choose $k = k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{4.9}$$

Moreover, since the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\overrightarrow{p}(.)}(\Omega)$, then there exists a subsequence still denoted $(T_k(u_n))_n$ such that

$$T_k(u_n) \rightharpoonup \eta_k \quad \text{weakly in } W_0^{1,\overrightarrow{p}(.)}(\Omega) \quad \text{as } n \to \infty, \quad \forall k > 0$$

and by the compact embedding, we have

$$T_k(u_n) \to \eta_k \quad \text{strongly in } L^{p^-(.)}(\Omega) \quad \text{and a.e. in } \Omega \quad \forall k > 0.$$

Consequently, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure. Thus,

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3} \quad \text{for all} \quad m, n \geq n_0(\delta, \varepsilon). \tag{4.10}$$

Finally, from (4.8),(4.9) and (4.10), we obtain that

$$\forall \delta, \varepsilon > 0, \ \exists n_0 = n_0(\delta, \varepsilon), \quad \text{such that} \quad \text{meas}\{|u_n - u_m| > \delta\} \leq \varepsilon, \quad \forall m, n \geq n_0.$$

Then $(u_n)_n$ is a Cauchy sequence in measure in Ω, then there exists a subsequence denoted by $(u_n)_n$ such that u_n converges to a measurable function u a.e. in Ω and

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,\overrightarrow{p}(.)}(\Omega) \quad \text{and a.e. in } \Omega \quad \forall k > 0 \tag{4.11}$$

It remains to prove that

$$\lim_{n \to +\infty} \sum_{i=1}^{N} \int_\Omega \mathcal{I}_{h,n}^0 dx = 0. \tag{4.12}$$

where

$$\mathcal{I}_{h,n}^0 = \left(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\right)\left(D_i T_k(u_n) - D_i T_k(u)\right).$$

Let $h > 0$, taking $v = u_n + T_1(u_n - T_h(u_n))^-$ as a test function in (3.1), and using the fact that $D_i T_1(u_n - T_h(u_n))^- = 0$ on the set $\{u_n < h\} \cup \{u_n > 1 + h\}$, we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\{-(h+1) \leq u_n \leq -h\}} a_i(x, u_n, \nabla u_n) D_i u_n dx \\ &\quad + \sum_{i=1}^{N} \int_{\{-(h+1) \leq u_n \leq -h\}} \phi_i^n(u_n) D_i u_n dx \\ &\leq - \int_\Omega f_n T_1(u_n - T_h(u_n))^- dx. \end{aligned} \tag{4.13}$$

We set $\Phi_i^n(s) = \int_0^s \phi_i^n \chi_{\{-(h+1)\leq t\leq -h\}} dt$, then $\Phi_i^n(0) = 0$ and $\Phi_i^n \in C^1(\mathbb{R})$, in view of the Green formula, we obtain

$$\sum_{i=1}^{N} \int_{\{-(h+1)\leq u_n\leq -h\}} \phi_i^n(u_n) D_i u_n dx = \sum_{i=1}^{N} \int_\Omega D_i \Phi_i^n(u_n) dx = \sum_{i=1}^{N} \int_{\partial\Omega} \Phi_i^n(u_n).\overrightarrow{n} d\sigma = 0, \tag{4.14}$$

since $u_n = 0$ on $\partial\Omega$, with $\overrightarrow{n} = (n_1, ..., n_N)$ the normal vector on $\partial\Omega$. Then, (4.13) and (4.14) implies

$$\sum_{i=1}^{N} \int_{\{-(h+1)\leq u_n\leq -h\}} a_i(x, u_n, \nabla u_n) D_i u_n dx \leq - \int_\Omega f_n T_1(u_n - T_h(u_n))^- dx. \tag{4.15}$$

By Lebesgue's theorem, we have

$$\lim_{h\to+\infty} \limsup_{n\to+\infty} \int_\Omega f_n T_1(u_n - T_h(u_n))^- dx = 0. \tag{4.16}$$

Combining (4.15) and (4.16), we obtain

$$\lim_{h\to+\infty} \lim_{n\to+\infty} \limsup \sum_{i=1}^{N} \int_{\{-(h+1)\leq u_n\leq -h\}} a_i(x, u_n, \nabla u_n) D_i u_n dx = 0. \tag{4.17}$$

Similarly, taking $v = u_n - \lambda T_1(u_n - T_h(u_n))^+$ as test function in approximate problem (3.1), we get

$$\lim_{h\to+\infty} \lim_{n\to+\infty} \limsup \sum_{i=1}^{N} \int_{\{h\leq u_n\leq h+1\}} a_i(x, u_n, \nabla u_n) D_i u_n dx = 0. \tag{4.18}$$

We consider the following function of one real variable

$$\varphi_h(s) = \begin{cases} 1 & \text{if } |s| \leq h, \\ 0 & \text{if } |s| \geq h+1, \\ h+1-|s| & \text{if } h \leq |s| \leq h+1. \end{cases}$$

Let $v = u_n - \lambda(T_k(u_n) - T_k(u))^+\varphi_h(u_n)$, where $h > k$ as test function in approximate problem (3.1), we obtain

$$\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)D_i(T_k(u_n)-T_k(u))^+\varphi_h(u_n)dx \tag{4.19}$$
$$+\underbrace{\sum_{i=1}^{N}\int_{\{h\le|u_n|\le h+1\}} a_i(x,u_n,\nabla u_n)(T_k(u_n)-T_k(u))^+D_iu_ndx}_{\mathcal{I}^1_{n,h}}$$
$$+\underbrace{\sum_{i=1}^{N}\int_\Omega \phi_i^n(u_n)D_i(T_k(u_n)-T_k(u))^+\varphi_h(u_n)dx}_{\mathcal{I}^2_{n,h}}$$
$$+\underbrace{\sum_{i=1}^{N}\int_\Omega \phi_i^n(u_n)(T_k(u_n)-T_k(u))^+D_iu_n\varphi_h'(u_n)dx}_{\mathcal{I}^3_{n,h}}$$
$$\le \underbrace{\int_\Omega f_n(T_k(u_n)-T_k(u))^+\varphi_h(u_n)dx}_{\mathcal{I}^4_{n,h}}\,.$$

By (4.17) and (4.18), we have

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\mathcal{I}^1_{n,h}=0. \tag{4.20}$$

Since $\varphi_h(u_n)=0$ if $|u_n|>h+1$, we have

$$\mathcal{I}^2_{h,n}=\sum_{i=1}^{N}\int_\Omega \phi_i^n(T_{h+1}(u_n))D_i(T_k(u_n)-T_k(u))^+\varphi_h(u_n)dx. \tag{4.21}$$

Using Lebesgue's theorem, we have $\phi_i^n(T_{h+1}(u_n))\varphi_h(u_n)\to\phi_i(T_{h+1}(u))\varphi_h(u)$ strongly in $L^{p_i'(\cdot)}(\Omega)$, and (4.11), (4.21) then

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\mathcal{I}^2_{n,h}=0\quad\text{and}\quad\lim_{h\to+\infty}\lim_{n\to+\infty}\mathcal{I}^3_{n,h}=0. \tag{4.22}$$

By another application of Lebesgue's theorem and (4.11), we obtain

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\mathcal{I}^4_{n,h}=0. \tag{4.23}$$

From (4.19)-(4.23) we conclude that,

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)D_i(T_k(u_n)-T_k(u))^+\varphi_h(u_n)dx\le 0,$$

which implies that

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|\leq k\}} a_i(x,u_n,\nabla u_n)D_i(T_k(u_n)-T_k(u))\varphi_h(u_n)dx$$

$$-\lim_{h\to+\infty}\lim_{n\to+\infty}\underbrace{\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|\leq k\}} a_i(x,u_n,\nabla u_n)D_iT_k(u)\varphi_h(u_n)dx}_{\mathcal{I}^5_{h,n}}\leq 0. \tag{4.24}$$

Since $\varphi_h(u_n)=0$ in $|u_n|>h+1$, we have

$$\mathcal{I}^5_{h,n}=\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|\leq k\}} a_i(x,T_{h+1}(u_n),\nabla T_{h+1}(u_n))D_iT_k(u)\varphi_h(u_n)dx. \tag{4.25}$$

Moreover, $(a_i(x,T_{h+1}(u_n),\nabla T_{h+1}(u_n)))_n$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, we have $a_i(x,T_{h+1}(u_n),\nabla T_{h+1}(u_n))$ converges to X_i^h weakly in $L^{p_i'(\cdot)}(\Omega)$. Then (4.25) implies

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\mathcal{I}^5_{h,n}=\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\{|u|>k\}} X_i^h D_iT_k(u)\varphi_h(u)dx=0. \tag{4.26}$$

By (4.24) and (4.26), we get (since $h>k$)

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a_i(x,T_k(u_n),\nabla T_k(u_n))D_i(T_k(u_n) \\ -T_k(u))\varphi_h(u_n)dx\leq 0. \tag{4.27}$$

Moreover, we have $a_i(x,T_k(u_n),\nabla T_k(u))\varphi_h(u_n)\rightharpoonup a_i(x,T_k(u),\nabla T_k(u))\varphi_h(u)$ strongly in $L^{p_i'(\cdot)}(\Omega)$ and (4.11), then

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a_i(x,T_k(u_n),\nabla T_k(u))D_i(T_k(u_n) \\ -T_k(u))\varphi_h(u_n)dx=0. \tag{4.28}$$

Combining (1.4), (4.27) and (4.28), we deduce

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0\}}\mathcal{I}^0_{h,n}\varphi_h(u_n)dx=0. \tag{4.29}$$

Similarly, we take $v=u_n+(T_k(u_n)-T_k(u))^-\varphi_h(u_n)$ as test function in approximate problem (3.1), we obtain

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\leq 0\}}\mathcal{I}^0_{h,n}\varphi_h(u_n)dx=0. \tag{4.30}$$

Combining (4.29) and (4.30) we get

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_\Omega \mathcal{I}^0_{h,n}\varphi_h(u_n)dx = 0. \tag{4.31}$$

Now, we prove

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_\Omega \mathcal{I}^0_{h,n}(1-\varphi_h(u_n))dx = 0. \tag{4.32}$$

Let $v = u_n + T_k(u_n)^-(1-\varphi_h(u_n))$ as test function in approximate problem (3.1), we obtain

$$\begin{aligned}
&-\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)D_iT_k(u_n)^-(1-\varphi_h(u_n))dx \qquad (4.33)\\
&\quad+\underbrace{\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)T_k(u_n)^-D_iu_n\varphi_h'(u_n)dx}_{\mathcal{I}^6_{h,n}}\\
&\quad-\underbrace{\sum_{i=1}^{N}\int_\Omega \phi_i^n(u_n)D_iT_k(u_n)^-(1-\varphi_h(u_n))dx}_{\mathcal{I}^7_{h,n}}\\
&\quad+\underbrace{\sum_{i=1}^{N}\int_\Omega \phi_i^n(u_n)T_k(u_n)^-D_iu_n\varphi_h'(u_n)dx}_{\mathcal{I}^8_{h,n}}\\
&\le -\underbrace{\int_\Omega f_nT_k(u_n)^-(1-\varphi_h(u_n))dx}_{\mathcal{I}^9_{h,n}}.
\end{aligned}$$

By (4.17) and (4.18), we have

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\mathcal{I}^6_{n,h} = 0. \tag{4.34}$$

Since $D_iT_k(u_n)^- \rightharpoonup D_iT_k(u)^-$ weakly in $L^{p_i(\cdot)}(\Omega)$ and $\phi_i^n(T_k(u_n))(1-\varphi_h(u_n)) \to \phi_i(T_k(u))(1-\varphi_h(u))$ strongly in $L^{p_i'(\cdot)}(\Omega)$, and By Lebesgue's theorem, we get

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\mathcal{I}^7_{n,h} = \lim_{h\to+\infty}\sum_{i=1}^{N}\int_\Omega \phi_i(T_k(u))D_iT_k(u)^-(1-\varphi_h(u))dx = 0. \tag{4.35}$$

We set $\Phi_i^n(s) = \int_0^t \phi_i^n(t)T_k(t)^-\varphi_h(t)dt$, by Green's Formula, we have

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\mathcal{I}^8_{n,h} = \lim_{h\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_\Omega D_i\Phi_i^n(u_n)dx = 0. \tag{4.36}$$

By the Lebesgue's theorem and (4.11), we obtain

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\mathcal{I}^9_{n,h}=0. \tag{4.37}$$

From (4.33)-(4.37) we conclude that,

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^N\int_{\{u_n\leq 0\}}a_i(x,u_n,\nabla u_n)D_iT_k(u_n)(1-\varphi_h(u_n))dx=0 \tag{4.38}$$

On the other hand, we take $v=u_n-\lambda T_k(u_n^+-\psi^+)(1-\varphi_h(u_n))$ as test function in approximate problem (3.1), we obtain

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega a_i(x,u_n,\nabla u_n)D_iT_k(u_n^+-\psi^+)(1-\varphi_h(u_n))dx \\
&\quad-\sum_{i=1}^N\int_\Omega a_i(x,u_n,\nabla u_n)T_k(u_n^+-\psi^+)D_iu_n\varphi_h'(u_n)dx \\
&\quad+\sum_{i=1}^N\int_\Omega \phi_i^n(u_n)D_iT_k(u_n^+-\psi^+)(1-\varphi_h(u_n))dx \\
&\quad-\sum_{i=1}^N\int_\Omega \phi_i^n(u_n)T_k(u_n^+-\psi^+)D_iu_n\varphi_h'(u_n)dx \\
&\leq\int_\Omega f_nT_k(u_n^+-\psi^+)(1-\varphi_h(u_n))dx.
\end{aligned} \tag{4.39}$$

Using young's inequality, we obtain

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega a_i(x,u_n,\nabla u_n)D_iT_k(u_n^+-\psi^+)(1-\varphi_h(u_n))dx \\
&\quad+\underbrace{\sum_{i=1}^N\int_\Omega \phi_i^n(u_n)T_k(u_n^+-\psi^+)D_iu_n\varphi_h'(u_n)dx}_{\mathcal{I}^{10}_{n,h}} \\
&\leq\underbrace{\sum_{i=1}^N\int_{\{-(h+1)\leq u_u\leq -h\}}a_i(x,u_n,\nabla u_n)D_iu_nT_k(u_n^+-\psi^+)dx}_{\mathcal{I}^{11}_{n,h}} \\
&\quad+\underbrace{\int_\Omega f_nT_k(u_n^+-\psi^+)(1-\varphi_h(u_n))dx}_{\mathcal{I}^{12}_{n,h}} \\
&\quad+\underbrace{\sum_{i=1}^N\int_{\{u_n^+-\psi^+\leq k\}}\phi_i^n(u_n)D_iu_n^+(1-\varphi_h(u_n))dx}_{\mathcal{I}^{13}_{n,h}}
\end{aligned}$$

$$+\underbrace{\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\le k\}}\phi_i^n(u_n)D_i\psi^+(1-\varphi_h(u_n))dx}_{\mathcal{I}_{n,h}^{14}}.$$

Similarly as (4.34), (4.36) and (4.37), we obtain

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\mathcal{I}_{n,h}^{11}=0,\quad \lim_{h\to+\infty}\lim_{n\to+\infty}\mathcal{I}_{n,h}^{10}=0\ \text{ and }\ \lim_{h\to+\infty}\lim_{n\to+\infty}\mathcal{I}_{n,h}^{12}=0. \tag{4.40}$$

Since

$$\mathcal{I}_{n,h}^{13}=\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\le k\}}\phi_i^n(T_{k+\|\psi^+\|_\infty}(u_n))D_iT_{k+\|\psi^+\|_\infty}(u_n^+)(1-\varphi_h(u_n))dx,$$
$$\mathcal{I}_{n,h}^{14}=\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\le k\}}\phi_i^n(T_{k+\|\psi^+\|_\infty}(u_n))D_i\psi^+(1-\varphi_h(u_n))dx,$$

since $D_iT_{k+\|\psi^+\|_\infty}(u_n^+)\rightharpoonup D_iT_{k+\|\psi^+\|_\infty}(u^+)$ weakly in $L^{p_i(\cdot)}(\Omega)$,

$$\phi_i^n(T_{k+\|\psi^+\|_\infty}(u_n))(1-\varphi_h(u_n))\to\phi_i(T_{k+\|\psi^+\|_\infty}(u))(1-\varphi_h(u))$$

strongly in $L^{p_i'(\cdot)}(\Omega)$ and ψ^+ bounded in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we have

$$\lim_{n\to+\infty}\mathcal{I}_{n,h}^{13}=\sum_{i=1}^{N}\int_{\{u^+-\psi^+\le k\}}\phi_i(T_{k+\|\psi^+\|_\infty}(u))D_iT_{k+\|\psi^+\|_\infty}(u^+)(1-\varphi_h(u))dx,$$
$$\lim_{n\to+\infty}\mathcal{I}_{n,h}^{14}=\sum_{i=1}^{N}\int_{\{u^+-\psi^+\le k\}}\phi_i(T_{k+\|\psi^+\|_\infty}(u))D_i\psi^+(1-\varphi_h(u))dx.$$

By Lebesgue's theorem, we have

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\mathcal{I}_{n,h}^{13}=0\quad\text{and}\quad\lim_{h\to+\infty}\lim_{n\to+\infty}\mathcal{I}_{n,h}^{14}=0. \tag{4.41}$$

From (**??**)-(4.41) we conclude that,

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\{u_n\ge0\}}a_i(x,u_n,\nabla u_n)D_iT_k(u_n)(1-\varphi_h(u_n))dx=0 \tag{4.42}$$

Combining (4.38) and (4.42), we get

$$\lim_{h\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\Omega}a_i(x,u_n,\nabla u_n)D_iT_k(u_n)(1-\varphi_h(u_n))dx=0 \tag{4.43}$$

Furthermore, we have

$$\sum_{i=1}^{N}\int_{\Omega}\mathcal{I}_{h,n}^{0}=\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n))$$
$$-a_i(x,T_k(u_n),\nabla T_k(u)))\left(D_iT_k(u_n)-D_iT_k(u)\right)\varphi_h(u_n)dx$$
$$+\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_k(u_n),\nabla T_k(u_n))D_iT_k(u_n)(1-\varphi_h(u_n))dx$$
$$-\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_k(u_n),\nabla T_k(u_n))D_iT_k(u)(1-\varphi_h(u_n))dx$$
$$-\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_k(u_n),\nabla T_k(u))(D_iT_k(u_n)-D_iT_k(u))(1-\varphi_h(u_n))dx.$$

By (4.31) and (4.43), the first and the second integrals on the right hand side converge to zero as n and h tend to $+\infty$. Since $(a_i(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $L^{p_i'(\cdot)}(\Omega)$ and $D_iT_k(u)(1-\varphi_h(u_n))$ converge to zero in $L^{p_i(\cdot)}(\Omega)$ as n and h tend to $+\infty$, hence the third integral on the right hand side converge to zero as n and h tend to $+\infty$. So, since $a_i(x,T_k(u_n),\nabla T_k(u_n))(1-\varphi_h(u_n))$ converges to $a_i(x,T_k(u),\nabla T_k(u))(1-\varphi_h(u))$ strongly in $L^{p_i(\cdot)}(\Omega)$ and $D_iT_k(u_n)\rightharpoonup D_iT_k(u)$ weakly in $L^{p_i(\cdot)}(\Omega)$, we obtain the fourth integral on the right hand side converge to zero as n and h tend to $+\infty$. Then, we get (4.12). Using (4.11), (4.12) and Lemma 2.5, we obtain (4.6). □

4.3. **Passing to the limit.** Now, let $w\in K_\psi\cap L^\infty(\Omega)$, we take $v=u_n-T_k(u_n-w)$ as test function in approximate problem (3.1), we obtain

$$\sum_{i=1}^{N}\int_{\Omega}[a_i(x,u_n,\nabla u_n)D_iT_k(u_n-w)dx+\int_{\Omega}\phi_i(u_n)D_iT_k(u_n-w)]dx$$
$$\le\int_{\Omega}fT_k(u_n-w)dx,$$

which implies that

$$\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_{k+\|w\|_\infty}(u_n),\nabla T_{k+\|w\|_\infty}(u_n))D_iT_k(u_n-w)dx$$
$$+\int_{\Omega}\phi_i(T_{k+\|w\|_\infty}(u_n))D_iT_k(u_n-w)dx$$
$$\le\int_{\Omega}fT_k(u_n-w)dx.$$

Since $T_k(u_n)\to T_k(u)$ strongly in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and a. e. in Ω $\forall k>0$, we have $a_i(x,T_{k+\|w\|_\infty}(u_n),\nabla T_{k+\|w\|_\infty}(u_n))\rightharpoonup a_i(x,T_{k+\|w\|_\infty}(u),\nabla T_{k+\|w\|_\infty}(u))$ weakly in $L^{p_i'(\cdot)}(\Omega)$, $\phi_i(T_{k+\|w\|_\infty}(u_n))\to\phi_i(T_{k+\|w\|_\infty}(u))$ strongly in $L^{p_i'(\cdot)}(\Omega)$ and $D_iT_k(u_n-w)\to D_iT_k(u-w)$ strongly in $L^{p_i(\cdot)}(\Omega)$ we can pass to limit in (3.1) this completes the proof of theorem 2.7.

References

[1] A. Kovacik, J. Rakosnik, *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*. Czechoslovak Math. J. 41 (1991), 592-618.

[2] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, 1969.

[3] X. Fan, *Anisotropic variable exponent Sobolev spaces and $\overrightarrow{p}(x)-$ Laplacian equations.* Complex. Var. Elliptic. Eq. 46 (2011), 623-642.

[4] X. Fan, J. Shen and D. Zhao, *Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$*. J. Math. Anal. Appl. 262 (2001), 749-760.

[5] H. Khelifi, F. Mokhtari, *Nonlinear degenerate anisotropic elliptic equations with variable exponents and L^1 data*. J. Part. Diff. Eq. 33 (2020), 1-16.

[6] F. Andreu, J.M. Mazòn, S. Segura De Léon and J. Toledo, *Quasi-linear elliptic and parabolic equations in L^1 with nonlinear boundary conditions*. Adv. Math. Sci. Appl. 7 (1997), 183-213.

[7] F. Andreu, N. Igbida, J.M. Mazòn and J. Toledo, *L^1 existence and uniqueness results for quasi-linear elliptic equations with nonlinear boundary conditions*. Ann. Ins. H. Poincaré Anal. Non Linéaire. 20 (2007), 61-89.

[8] I. Ibrango, *Entropy Solution For Doubly Nonlinear Elliptic Anisotropic Problems With Fourier Boundary Condition*. Differ. Inclusions. Control. Optim. 35 (2015), 123-150.

[9] M. Mihailescu, P. Pucci and V. Radulescu, *Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent*. J. Math. Anal. Appl. 340 (2008), 687-698.

[10] M.B. Benboubker, E. Azroul and A. Barbara, *Quasilinear elliptic problems with nonstandard growths*. Elec. J. Diffe. Equa. 62 (2011), 1-16.

[11] M.B. Benboubker, H. Chrayteh, H. Hjiaj and Y. Chihab, *Existence of solutions in the sense of distributions of anisotropic nonlinear elliptic equations with variable exponent*. Topol. Methods. Nonlinear. Anal. 46 (2015), 665-693.

[12] M.B. Benboubker, H. Hjiaj and S. Ouaro, *Entropy solutions to nonlinear elliptic anisotropic problem with variable exponent*. Appli. Anal. Comp. 4 (2014), 245-270.

[13] Y. Chen, M. Levine Srao, *Variable exponent, linear growth functionals in image restoration*. SIAM J Appl Math. 66 (2006), 1383-1406.

[14] H.B. Keller and D.S. Chohen, *Some positive problems suggested by nonlinear heat generators*. J.Math. Mech. 16 (1967), 1361-1376.

[15] A. Ambrosetti, H. Brézis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*. J.Funct. Anal. 122 (1994), 519-543. .

[16] Y. Akdim and A. Salmani, *Existence results for nonlinear anisotropic elliptic unilateral prolems*. AIP. Conference. Proceedings. 2074 (2019), id.020008.

[17] Y. Akdim and A. Salmani, *solvability of nonlinear anisotropic elliptic unilateral problems with Variable exponents*. Gulf J. Math. 4 (2018), 133-150.

[18] Y. Akdim, C. Allalou and A. Salmani, *Existence of solutions for some nonlinear elliptic anisotropic unilateral problems with lower prder terms*. Moroccan J. of Pure and Appl. Anal. 4 (2018), 171-188.

Department of Mathematics and Informatics, University of Algiers, Algiers, Algeria. 2 Street Didouche Mourad Algiers.

E-mail address: khelifi.hichemedp@gmail.com