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CONVERGENCE THEOREMS OF TWO MULTIVALUED MAPPINGS SATISFYING THE JOINTLY DEMICLOSEDNESS PRINCIPLE IN BANACH SPACES

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Abstract. In this paper, we propose and study two iteration schemes (modified Halpern's type and HS-iteration schemes). It is proved that if two multivalued quasi-nonexpansive mappings are of type one and satisfy the jointly demiclosedness principle, the iterative sequences generated in our schemes converge to a common fixed point of the mappings without the condition that the common fixed point set is strict. Our main results improve, generalize and extend the corresponding results announced recently literature from single-valued mappings to multivalued mappings.

1. Introduction

Let E be a Banach space and D a nonempty closed convex subset of E. Throughout the paper, N, $CB(E), KC(E)$ and $\mathcal{P}(E)$ denote the set of positive integers, the family of closed ad bounded subsets of E, the family of nonempty compact convex subsets of E and the family of proximinal subsets of E, respectively. A subset D is called a proximinal if, for each $x \in E$, there exists a point $m \in D$ such that

$$d(x, m) = \inf\{\|x - y\| : y \in D\} = d(x, D). \tag{1.1}$$

It is well known that weakly compact convex subsets of a Banach space and closed convex subsets of a uniformly convex Banach spaces are proximinal. Let $Q : D \longrightarrow D$ be a single-valued mapping of D into itself and $S, T : D \longrightarrow CB(D)$

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be two multivalued mappings of D into $CB(D)$. The set of fixed points of Q, S and T will be denoted by $F(Q) = \{x \in D : Qx = x\}$, $F(S) = \{x \in E : x \in Sx$ and $F(T) = \{x \in E : x \in Tx$, respectively. A point x is a common fixed point of S and T if $x \in F(S) \cap F(T)$.

In 1965, Browder [31] established the first fixed point theorem for single-valued nonexpansive self mappings. More precisely, he proved that if C is a bounded closed convex subset of a Hilbert space H and T is a nonexpansive mapping of C into itself, then T has a fixed point in C. Later, both Browder [32] and Gohde [33] simultaneously proved that the same is true if E is a uniformly convex Banach space. Kirk [34] also proved the following theorem:

Theorem 1.1. *Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of C into itself. Then F(T) is nonempty.*

After Kirk's theorem, many fixed point theorems concerning single-valued mappings have been proved in a Hilbert space or a Banach space (see, e.g., [2], [4], [6], [8]-[10] and the references contained in them). In particular, Baillon and Schoneberg [35] introduced the concept of asymptotic normal structure and generalized Kirk's fixed point theorem as follows:

Theorem 1.2. *Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has asymptotic normal structure. Let T be a nonexpansive mapping of C into itself. Then F(T) is nonempty.*

In recent times, the study of fixed points of multivalued mappings has stimulated the interest of a good number of well known mathematicians across the world (see [9]-[24], and the references contained in them). Their motivation possibly comes from the usefulness of fixed point theory in real life applications such as in Game Theory and Market Economy, and in some other area of mathematics such as Nonsmooth Differential equations, Optimization Theory and Differential Inclusion. The following are the connection of fixed point theory of multivalued mapping and these applications:

Game Theory and Market Economy [37]. In game theory and market economy, the existence of equilibrium was uniformly obtained by the application of fixed point theorem. More precisely, under some regularity conditions, given any game, there always exists a multivalued mapping whose fixed point coincide with equilibrium points of the game.An illustrative example of such application is the Nash equilibrium theorem [50]. Consider a game $G = (v_n, K_n)$ with N players denoted by $n = 1, 2, \cdots, N$, where $K_n \subset R^n$ is the set of possible strategies of the n^{th} player and is assumed to be nonempty, compact and convex, and $v_n : K = K_1 \times K_2 \times \cdots \times K_N \longrightarrow R$ is the game function (or payoff) of the player n and is assumed to be continuous. The player n can take individual actions represented by a vector $\sigma_n \in K_n$. All players together can take a collective action, which is a combined vector $\sigma = \sigma_1, \sigma_2, \cdots, \sigma_N$. For each $n, \sigma \in K$ and $w_n \in K_n$, we use the following standard notations:

$$\begin{aligned} K_{-n} &= K_1 \times K_2 \times K_{n-1} \times \cdots \times K_N \\ \sigma_{-n} &= \sigma_1, \sigma_2, \cdots, \sigma_{n-1}, \sigma_{n+1} \cdots, \sigma_N \\ (w_n, \sigma_{-n}) &= \sigma_1, \sigma_2, \cdots, \sigma_{n-1}, w_n, \sigma_{n+1} \cdots, \sigma_N. \end{aligned}$$

A strategy $\bar{\sigma_n} \in K_n$ permits the n^{th} player maximise his game under the condition that the remaining players have chosen their strategies σ_{-n} if and only if $v_n(\bar{\sigma_n}, \sigma_{-n}) = max_{w_n \in K_n} v_n(w_n, \sigma_{-n})$. Now, let $T_n : K_{-n} \longrightarrow 2^K$ be the multivalued mapping defined by $T_n(\sigma_{-n}) = \arg\max_{w_n \in K_n} v_n(w_n, \sigma_{-n})$, $\forall \sigma_{-n} \in K_{-n}$.

Definition 1.3. A collective action $\bar{\sigma} = (\bar{\sigma_1}, \bar{\sigma_2}, \cdots, \bar{\sigma_N}) \in K$ is called a Nash equilibrium point if, for each $n, \bar{\sigma_n}$ is the best response for the n^{th} player to the action $\bar{\sigma_n}$ made by the remaining players. That is, for each n, $v_n(\bar{\sigma} = max_{w_n \in K_n} v_n(w_n, \sigma_{-n})$ or equivalently, $\bar{\sigma_n} \in T_n(\bar{\sigma_{-n}})$. This is equivalent to say that $\bar{\sigma}$ is a fixed point of the multivalued mapping $T : K \longrightarrow 2^K$ defined by

$$T(\sigma) = T_1(\sigma_{-1}) \times T_1(\sigma_{-2}) \cdots \times T_1(\sigma_{-N}) \tag{1.2}$$

From the point of view of social recognition, game theory is perhaps the most successful area of application of fixed point theory of multivalued mappings. However, it has been remarked that the applications of this theory to equilibrium are mostly static: they enhance understanding conditions under which equilibrium may be achieved but do not indicate how to construct a process starting from a non-equilibrium point and convergent to equilibrium solution. One instance of this could be seen below:

Game Theory and Shakespeare's play [38] Othello is a black General who marries Desdemona. He is envied by Iago whom he has complete trust on (his tragic flaw). Envious Iago plans to destroy, thus the game starts. Iago rebels the Desdemona's family against Othello and he suspects Othello to doubt his wife's chastity. He even provides false evidence of Desdemona's betrayal. Othello kills Desdemona and destroys himself as well. Iago wish to destroy Othello out of hatred for him. His preference would be, ruining Othello's happy life (by murdering his faithful wife by his own hands). Suspected Othello prefers to maintain his reputation by murdering his unfaithful wife. Therefore, Iago has two choices to either suspect Othello or not to suspect. Othello also in return has two choices of either murdering his wife or not murdering her. The case above [38] is part of the problem that is being addressed by iterative methods for fixed point of multivalued mappings.

Optimization Problems with Constraints [36].

Let H be a real Hilbert space and $f : H \longrightarrow R \cup \{+\infty\}$ be a proper convex lower semicontinuous function and $\psi : H \longrightarrow 2^H$ be a multivalued mapping. Consider the following optimization problem:

$$(P) = \begin{cases} min f(x), \\ 0 \in \psi(x). \end{cases}$$

It is known that the multivalued map, ∂f, the subdifferential of f is a maximal monotone (see [49] for details), where for $x, w \in H$,

$$w \in \partial f(x) \Leftrightarrow f(y) - f(x) \geq \langle y - x, w \rangle, \forall y \in H;$$

$$\Leftrightarrow x \in \arg\min(f - \langle , .w \rangle.$$

It is easily seen that, for $x \in H$, with $0 \in \psi(x), x$ is a solution of (P) if and only if $0 \in \partial f(x) \cap \psi(x)$, or equivalently $x \in T_1 \cap T_2$, with $T_1 = I - \partial f$ and $T_2 = I - \psi$, where I is the identity map of H. Therefore, x is a solution of (P) if and only if x is a common fixed point of the multivalued mappings T_1 and T_2. Let H be a Hausdorff metric induced by the metric d of E, that is,

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}, \tag{1.3}$$

for every $A, B \in CB(E)$. In the sequel, the following definitions will be needed:

Definition 1.4. Recall that a single-valued mapping $Q : D \longrightarrow D$ is called nonexpansive [1] if

$$\|Qx - Qy\| \leq \|x - y\|, \forall x, y \in D. \tag{1.4}$$

A subset D of E is said to be a retract of E if there exists a continuous mapping $Q : E \longrightarrow D$ (called retraction) such that $Qx = x$ for all $x \in E$. A retraction Q from D onto E is called sunny if the following property holds: $Q(Qx + t(x + Qx)) = Qx$ for all $x \in D$ and $t \geq 0$ with $Qx + t(x + Qx) \in D$. A retract of a Hausdorff space must be a closed subset. It is well known that every closed convex subset of a uniformly convex Banach space E is a retract of E.

It has been established [3, Theorem 13.1] that in a smooth Banach space E, a retraction Q from D onto E is both sunny and nonexpansive if and only if

$$\langle x - Qx, J(y - Qx) \leq 0, \forall x \in D \quad \text{and} \quad y \in E. \tag{1.5}$$

Hence, there is at most one sunny nonexpansive retraction from D onto E. For example, if E is a nonempty, closed and convex subset of a Hilbert space H, then the nearest point projection P_E from E onto E is the unique sunny nonexpansive retraction of E onto E. This is not true in general for all Banach spaces, since outside Hilbert space, nearest point projections, although sunny are no longer nonexpansive. Conversely, sunny nonexpansive retraction do sometimes play a similar role in Banach space as the nearest point projections in Hilbert space. Thus, it becomes necessary to ask the following question:

Question 1.5. *Which subsets of a Banach space does a sunny nonexpansive retraction exists? If it does exists, how can one find it?*

It was proved [3,Theorem 13.2] that if C is a closed convex subset of a uniformly smooth Banach space and $T : C \longrightarrow C$ is nonexpansive, then the fixed point set is a sunny retraction of C. Bruck [2,Theorem 2] also proved that if C is a closed convex subset of a reflexive Banach space, every bounded, closed and convex subset of which has the fixed point property for nonexpansive mappings and $T : C \longrightarrow C$ is nonexpansive, then its fixed point set is a nonexpansive retraction of C.

Definition 1.6. Recall that a multivalued mapping $T : D \longrightarrow CB(D)$ is called:

(1) -Lipschitizian if there exists $L > 0$ such that
$$H(Tx, Ty) \leq L\|x - y\|, \forall x, y \in D. \tag{1.6}$$
Note that if $L \in (0, 1)$ in (1.6), then T is called a contraction; and if $L = 1$ in (1.6), then T is called nonexpansive. It is worth mentioning that the fixed points of multivalued contraction and nonexpansive mappings have been extensively studied in recent times by different researchers (see, e.g., [11-[19] and the references therein for more details).

(2) ultivalued quasi-nonexpansive if $F(T) = \{x \in D : x \in Tx\} \neq \emptyset$ and $\forall x \in D$, the inequality below holds:
$$H(Tx; Tq) \leq \|x - q\|, \forall q \in F(T). \tag{1.7}$$
Note that every multivalued nonexpansive mapping with a nonempty fixed point set is multivalued quasi-nonexpansive.

(3) onsreading-type [42] if
$$2H(Tx, Ty)^2 \leq d(Tx, y)^2 + d(x, Tx)^2, \forall x, y \in D, \tag{1.8}$$
where D is a subset of a Hilbert space H. It is easy to see that if T is nonspreading-type, then T is nonspreading in the case of single-valued mapping (see [40, 41]). Moreover, if T is nonspreading-type and $F(T) = \{x \in D : x \in Tx\} \neq \emptyset$, then T is quasi-nonexpansive. Note that nonspreading-type multivalued mappings and multivalued nonexpansive mappings are independent as could be seen in the example below:

Example 1.7. (see [42]) Let $C = C[-3, 3]$ with the usual norm. Define a multivalued mapping $T : C \longrightarrow CB(C)$ by
$$Tx = \begin{cases} 0, & \text{if} \quad x \in [-2, 2]; \\ \left[-\dfrac{|x|}{|x| + 1}, \dfrac{|x|}{|x| + 1} \right], & if \quad x \quad \text{is not in} \quad [-2, 2]. \end{cases}$$
It is shown in [42] that T is nonspreading-type multivalued mapping but not multivalued nonexpansive.

(4) losed if for any sequence $\{x_n\}_{n \in N} \subset D$ with $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then we have $Tx_0 = y_0$.

Due its connection with so many contractive-type mappings, several authors have studied iterative methods for approximating fixed points of nonexpansive and quasi-nonexpansive mappings (see, e.g., [1]-[3], [6], [8], [10], etc. and the references contained therein). ln 1953, Mann [43] introduced the following iteration scheme, which is referred to as Mann iteration method:
$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \tag{1.9}$$
where an initial guess $x_0 \in C$ is arbitrary and $\{\alpha_n\}_{n \in N} \subseteq [0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. The Mann iteration scheme has been extensively investigated (see, e.g., [44]). ln an infinitely dimensional Hilbert spaces, the Mann iteration sequence can only guarantee weak convergence (see [45]). To achieve strong convergence, different authors have modified the Mann iteration

method (see [39]) in many ways.

In 1967, Halpern [39] studied the following iteration scheme :

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \tag{1.10}$$

where $\{\alpha_n\}_{n\in N}$ is a real sequence in $[0,1]$ satisfying some appropriate conditions. He proved strong convergence of $\{x_n\}_{n\in N}$ to the fixed point of T, where $\alpha_n = n^a$, for $a \in (0,1)$, in the setting of Hilbert space. Recently, many researchers have used (1.10) in its original form, and the modified version, in approximating the fixed points of nonexpansive mappings and other classes of nonlinear mappings in different spaces (see [1], [46]-[47] and the references therein).

For approximating the fixed points of Lipschitz pseudocontractive mapping T, lshikawa [48] introduced the following algorithm, which is called lshikawa iteration algorithm:

$$\begin{cases} x_1 = x \in C; \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n; \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, n \geq 0 \end{cases} \tag{1.11}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions: $0 \leq \alpha_n \leq \beta_n \leq 1$; $\lim_{n\to\infty}\beta_n = 0$; $\sum_{n=1}^{\infty}\alpha_n\beta_n = \infty$. He showed that the sequence defined by (1.11) converges strongly to a fixed point of the mapping T provided C is a compact convex subset of a Hilbert space H.

Recently, Agrwal, O'Regan and Sahu [49] introduced the S-iteration algorithm in Banach space as follows:

$$\begin{cases} x_1 =\in K; \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n; \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, n \geq 1, \end{cases} \tag{1.12}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$. They showed that their algorithm is independent of (1.9) and (1.11) and converges faster than both (1.9) and (1.11) (see [50] for details).

Most recently, Naraghirad [1] introduced (1.13) as a generalization of (1.10) as follows:

$$\begin{cases} x_1 = x \in D; \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n; \\ y_n = (1-\beta_n)S_n x_n + \beta_n Tx_n, \end{cases} \tag{1.13}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$. They proved strong convergence of (1.13) to $Q_{\mathcal{F}}u$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction from E onto $\mathcal{F}$.

The demiclosedness principle, which was first studied by Opial [6], is one of the indispensable tools in proving weak and strong convergence theorems for both single-valued and multivalued nonlinear mappings. Notably, the theory of fixed points with the associated mappings satisfying demiclosedness principle due to Opial [6] has been deeply investigated for the past forty (40) years or so, and much more intensively recently (see, e.g, [1] and the references therein). Although some interesting results have been obtained, it is worth mentioning that, in some cases, the mapping T of the class of nonexpansive mappings defined in the setting of a Hilbert space H does not necessarily satisfies the demiclosedness principle due to Opial [6] (see [1], Example 2.1 for details). Consequently, it is natural to ask:

Question 1.8. *Is there any way one can obtain strong convergence theorems of Halpern's type for such mappings that fail to satisfy the original demiclosedness principle due to Opial in the setting of Banach spaces?*

Naraghrad [1] gave an affirmative answer to the above question using the idea of jointly demiclosedness principle (Recall that if C is a nonempty subset of of a Banach space E, then a pair $S, T : C \longrightarrow C$ satisfies jointly demiclosedness principle if $x_n \subset C$ converges weakly to a point $z \in C$ and $lim_{n\to\infty}\|Sx_n - Tx_n\| = 0$, then $Sz = z$ and $Tz = z$; that is, $S - T$ is jointly demiclosed at zero) which they introduced. More precisely, they prove the following theorem:

Theorem 1.9. *(EN) Let E be a Banach space and C a nonempty, closed and convex subset of E and $v \in C$. Let $S, T : C \longrightarrow C$ be two quasi-nonexpansive self mappings such that $\mathcal{F} = F(S) \cap F(T) \neq \emptyset$ is closed and convex. Let S, T satisfies jointly demiclosedness principle on C and fxng be the sequence defined by*

$$\begin{cases} x_1 = x \in D; \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n; \\ y_n = (1 - \beta_n) S x_n + \beta_n T x_n, \forall n \in N, \end{cases} \tag{1.14}$$

where $\{\alpha_n\}_{n\in N}$ and $\{\beta_n\}_{n\in N}$ are real sequences in $(0, 1)$. If the following conditions hold: $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=1}^{\infty} \alpha_n = \infty$; $0 < \liminf_{n\to\infty} \beta_n(1 - \beta_n) \leq \limsup_{n\to\infty} \beta_n(1 - \beta_n) < 1$. Then, the sequence defined (1.14) converges strongly to $Q_{\mathcal{F}} u$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction from E onto $\mathcal{F}$.

Remark 1.10. It is remarked in [1] that if $S = I$, where I is the identity mapping on E, then $I - T$ is demiclosed at zero. Again, if S and T satisfy the demiclosedness principle due to Opial [6], then (S, T) satisfies the jointly demiclosedness . Regrettably, the converse is not in general true as could be seen in [1, Example 2.1].

Example 1.11. (see [1]) Let $E = \ell^2(N)$, where

$$\ell^2(N) = \left\{ \sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n \cdots) : \sum_{n=1}^{\infty} \|\sigma_n\|^2 < \infty \right\}$$

$$\|\sigma\| = \left(\sum_{n=1}^{\infty} \|\sigma_n\|^2 \right)^{\frac{1}{2}}, \forall \sigma \in \ell^2(N),$$

$$(\sigma, \eta) = \sum_{n=1}^{\infty} \sigma_n \eta_n, \forall \sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n \cdots), \eta = (\eta_1, \eta_2, \cdots, \eta_n \cdots) \in \ell^2(N).$$

Let $\{y_n\}_{n\in N\cup\{0\}} \subset E$ be a sequence defined by

$$\begin{array}{ccl} y_0 & = & (1,0,0,0,0\cdots) \\ y_1 & = & (1,1,0,0,0,0,\cdots) \\ y_2 & = & (1,0,1,0,0,0,\cdots) \\ & \cdots\cdots\cdots & \\ y_n & = & (\sigma_{n,1},\sigma_{n,2}\cdots,\sigma_{n,k},\cdots) \\ & \cdots\cdots\cdots & \end{array}$$

where

$$\sigma_{n,k} = \begin{cases} 1, & \text{if} \quad k = 1, n+1, \\ 0, & \text{if} \quad k \neq 1, k \neq n+1, \end{cases}$$

Define two mappings $S, T : E \longrightarrow E$ by

$$S(y) = \begin{cases} \dfrac{-n}{n+1} y, & \text{if} \quad y = y_0, \\ -y, & \text{if} \quad y \neq y_0, \forall n \geq 0, \end{cases}$$

and

$$T(y) = \begin{cases} \dfrac{n}{n+1} y, & \text{if} \quad y = y_0, \\ y, & \text{if} \quad y \neq y_0. \end{cases}$$

Then, T does not satisfy the original demiclosedness principle but (S,T) does satisfies the jointly demiclosedness principle (see, e.g., [1] for details).

The idea of approximating fixed points for multivalued contraction and nonexpansive mappings using the Hausdorff metric was initiated by Markin [12]. Thereafter, various iteration schemes have been developed and used to approximate the fixed points of multivalued nonexpansive mappings in Banach spaces by different authors (see, e.g., [9], [15]-[24] and the reference therein); and their relentless efforts led to interesting results in the study fixed point theory with real applications in convex optimization, control theory, economics, differential inclusion and related topics. In 1974, Lim [16], using the Edelstein's method of asymtotic center (see [19] for details), proved that every multivalued nonexpansive self mapping $T : E \to K(E)$ has a fixed point, where E is a nonempty bounded closed convex subset of a uniformly convex Banach space. ln 1998, Kirk and Massa [15] extended Lim's theorem to assure the existence of fixed point of multivalued nonexpansive self mapping $T : E \longrightarrow K(E)$, where E is nonempty closed convex subset of a Banach X space which has the property that asymptotic center in E of each bounded sequence of X is nonempty and compact. ln 2001, Xu [17] extended Kirk and Massa's work to nonself mapping $T : E \longrightarrow K(X)$ which satisfies an inwardness condition. Let $T : C \longrightarrow P(C)$ be multivalued mapping. Define

$$P_T(x) = \{y \in Tx : d(x,Tx) = \|x-y\|\}, \forall x \in K. \tag{1.15}$$

Then, P_T is called best approximation operator. Approximating fixed points of multivalued mappings using best approximation method was first introduced by

Hussan and Khan [33] in 2003.They proved that if C is a closed unbounded subset of a Hilbert space H, then every $\star$-nonexpansive multivalued mapping has a fixed point. Since then, numerous results have been proved for multivalued mappings using best approximation operator in Banach spaces (see, for example, [20], [23], [24], etc. and the references therein). Recently, it has been observed that approximation of fixed points of multivalued mappings $T : D(T) \subseteq E \longrightarrow 2^E$ with regard to Hausdorff metric have been impossible without imposing the conditions that either the fixed point set of T is strict or T is a multivalued mapping for which P_T satisfies some contractive conditions. Consequently, it becomes natural to ask if there exists a class of multivalued mappings with nonempty fixed point set for which neither the fixed point set of T is strict nor P_T satisfies any contractive condition. Isiogugu [27] gave an affirmative answer to the above question and also noted that this class of mapping possesses other interesting properties of some existing maps recently studied in (see [20], [23], [24], etc. and the references therein and the references therein). In line with this observation, a new type of map, which she called type-one mapping (Recall that a multivalued map $S : D(S) \subseteq X \longrightarrow 2^X$ defined on a normed space E is called a type one if given any pair $r, g \in D(S)$, we have $\|v - v\| \leq \Phi(Sr, Sg), \forall u \in P_S r, v \in P_S g)$, was introduced and used in direct approximation of fixed points for multivalued mappings. Moltivated by these facts, we introduce an analog to (1.13) for multivalued mappings and also study a new iterative algorithm that is independent of (1.13) for single-valued mappings. The two iterative algorithms are defined as follows. Let D be a nonempty subset of a real Banach space E and $S, T : D \longrightarrow CB(D)$ be two multivalued mappings with $F(T) \neq \emptyset$. Compute the sequence $\{x_n\}_{n \in N}$ by the iterative schemes

$$\begin{cases} x_1 = x \in D; \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n; \\ y_n = (1 - \beta_n) v_n + \beta_n w_n, \forall n \in N, \end{cases} \tag{1.16}$$

and

$$\begin{cases} x_1 = x \in D; \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n; \\ y_n = (1 - \beta_n) v_n + \beta_n w_n, \forall n \in N, \end{cases} \tag{1.17}$$

where $\{\alpha_n\}_{n \in N}$ and $\{\beta_n\}_{n \in N}$ are real sequences in $(0, 1)$, $v_n \in Sx_n$ with $\|x_n - v_n\| = d(x_n, Sx_n)$, $w_n \in Tx_n$ with $\|x_n - w_n\| = d(x_n, Tx_n)$ and $z_n \in Ty_n$ with $\|y_n - z_n\| = d(y_n, Ty_n)$. It is important to note that for single-valued mapping T, the algorithm (1.17) reduces to (1.13). However, (1.16) is independent of (1.13). We also note that (1.16) and (1.17) are independent. The purpose of this paper is to establish strong convergence theorems of the iterative algorithms (1.16) and (1.17) for multivalued quasi-nonexpansive mappings in uniformly convex Banach space.

2. Preliminary

For the sake of convenience, we restate the following concepts and results. Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is a function $\delta_E(\varepsilon) : (0, 2] \longrightarrow (0, 2]$ defined by

$$\delta_E(\varepsilon) = \inf\{1 - \|\frac{1}{2}(x + y)\| : \|x\| = 1, \|y\| = 1, \varepsilon = \|x - y\|\}.$$

A Banach space E is uniformly convex if and if $\delta_E(\varepsilon) > 0$, for all $\varepsilon \in (0, 2]$.

Definition 2.1. A multivalued mapping $T : D \longrightarrow CB(D)$ is said to satisfy condition E_μ, where $\mu \geq 0$, if for each $x, y \in D$, $d(x, Ty) \leq \mu d(x, Tx) + \|x - y\|$. We say that T satisfies condition (E) whenever T satisfies (E_μ) for some $\mu \geq 0$.

Definition 2.2. The space E has Opial condition [6] if for any sequence $\{x_n\}$ in E, x_n converges to x weakly,it follows that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $x \neq y$.

Examples of Banach spaces satisfying Opial conditions are Hilbert spaces and all spaces $l^p (1 < p < \infty)$. On the other hand, $L^p[0, \pi]$ with $1 < p \neq 2$ fails to satisfy Opial condition.

Lemma 2.3 (1). *Let $\{s_n\}_{n\in N}$ be a sequence of nonnegative real numbers satisfyingthe inequality: $s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n\delta_n$, $\forall n \geq 1$, where $\{\gamma_n\}_{n\in N}$ and $\{\delta_n\}_{n\in N}$ satisfy the following conditions: $\{\gamma_n\}_{n\in N} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$, or equivalently, $\prod_{n=1}^{\infty}(1 - \gamma_n) = 0$; $\limsup_{n\to\infty} \delta_n \leq 0$ or $(ii)'$ $\sum_{n1}^{\infty} \gamma_n\delta_n < \infty$. Then, $\lim_{n\to\infty} s_n = 0$.*

Lemma 2.4. *(see [1]) Let X be a uniformly convex Banach space and let $r > 0$. Then, there exists a strictly increasing continous function $g : [0, \infty) \longrightarrow [0, \infty)$ with $g(0) = 0$ such that $\|\lambda x+(1-\lambda)y\|^2 \leq \lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)g(\|x-y\|)$ for all $x, y \in B_r = \{z \in X : \|z\| \leq r, \lambda \in [0, 1]$.*

Lemma 2.5. *(see [1]) Let E be a real Banach space and $J : E \longrightarrow 2^{E^\star}$ be the normalized duality mapping of E. Then,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle, \forall x, y \in E. \tag{2.1}$$

Lemma 2.6. *(see [1]) Let $\{a_n\}_{n\in N}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}_{n\in N}$ of $\{n\}_{n\in N}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in N$. Then, there exists a subsequence $\{m_k\}_{k\in N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) $k \in N$:*

$$a_{m_k} \leq a_{m_k+1} \quad \textit{and} \quad a_k \leq a_{m_k+1} \tag{2.2}$$

In fact, $m_k = maxj \leq k : a_j < a_j + 1$.

Lemma 2.7. *(see [1]) Let C and D be nonempty subsets of a Banach space E with $D \subset C$ and let $Q_D : C \longrightarrow D$ be a retraction from C into D. then, Q_D is sunny and nonexpansive if and only if*

$$\langle z - Q_D(z), J(y - Q_D(z))\rangle \leq 0, \forall z \in C \quad \textit{and} \quad \forall y \in D, \tag{2.3}$$

where J is the normalized duality mapping of E.

Lemma 2.8. *(see [6]) Let X be the Banach space which satisfies the Opial property and $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\|x_n - u\|$ and $\|x_n - v\|$ exists. If $\{x_{n_i}\}$ and $\{x_{n_j}\}$ are subsequences of $\{x_n\}$ which converges to u and v respectively, then $u = v$.*

3. Main Results

ln the sequel, we need the following definitions and lemmas to prove our main results.

Definition 3.1. Let D be a nonempty closed convex subset of a Banach space E. A pair (S, T) of multivalued mappings $S, T : D \longrightarrow CB(D)$ satisfies the jointly demiclosedness principle , in the sense of Naraghirad [1], if for any sequence $\{x_n\}_{\geq 1}$ converging weakly to a point $q \in D$ and there exist $u_n \in Sx_n$ and $v_n \in Tx_n$ with $\|u_n - v_n\| = d(Sx_n, Tx_n)$ such that $\|u_n - v_n\| \to 0$ as $n \to \infty$, then $Sq = q$ and $Tq = q$; that is, $S - T$ is jointly demiclosed at zero. In particular, if $S = I$, where I is the identity mapping on E, then $I - T$ is demiclosed at zero.

In [1], it was shown that if S and T satisfy the original demiclosedness principle, then (S, T) satisfies the jointly demiclosedness principle. Also, an example was given to show that the converse is not true in general.

Theorem 3.2. *Let E be a Banach space and D a nonempty, closed and convex subset of E and $v \in D$. Let $S, T : D \longrightarrow \mathcal{P}(D)$ be two multivalued type one quasi-nonexpansive multivalued self mappings from D into the family of proximinal subsets of D such that $\mathcal{F} = F(S) \cap F(T) \neq \emptyset$ is closed and convex.. Let S, T satifies jointly demiclosedness principle on D and $\{x_n\}$ be the sequence defined by*

$$\begin{cases} x_1 = x \in D; \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n; \\ y_n = (1 - \beta_n) v_n + \beta_n w_n, \forall n \in N, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$, $v_n \in Sx_n$ with $\|x_n - v_n\| = d(x_n, Sx_n)$, $w_n \in Tx_n$ with $\|x_n - w_n\| = d(x_n, Tx_n)$ and $z_n \in Ty_n$ with $\|y_n - z_{n,i}\| = d(y_n, Ty_n)$. If the following conditions hold:

i. $\lim_{n \to \infty} \alpha_n = 0$
ii. $\sum_{n=1}^{\infty} \alpha_n = \infty$;
iii. $0 < \liminf_{n \to \infty} \beta_n (1 - \beta_n) \leq \limsup_{n \to \infty} \beta_n (1 - \beta_n) < 1$.

Then, the sequence defined in (3.1) converges strongly to $Q_{\mathcal{F}} v$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retaction from E onto $\mathcal{F}$.

Proof. Since S and T are multivalued quasi-nonexpansive mappings, it follows that $\mathcal{F}$ is closed and convex. Set $z = Q_{\mathcal{F}} v$. Let $q \in \mathcal{F}$ be fixed. Using Lemma 2.2, we can find a strictly increasing function $g : [0, \infty) \longrightarrow [0, \infty)$ with $g(0) = 0$

such that the following estimates remain valid:

$$\begin{aligned} \|y_n - q\|^2 &= \|(1-\beta_n)(v_n - q) + \beta_n(w_n - q)\|^2 \\ &\le (1-\beta_n)\|v_n - q\|^2 + \beta_n\|w_n - q\|^2 - \beta_n(1-\beta_n)g(\|v_n - w_n\|) \\ &\le (1-\beta_n)\|x_n - q\|^2 + \beta_n\|x_n - q\|^2 - \beta_n(1-\beta_n)g(\|v_n - w_n\|) \\ &= \|x_n - q\|^2 - \beta_n(1-\beta_n)g(\|v_n - w_n\|) \qquad (3.2) \\ &\le \|x_n - q\|^2. \qquad (3.3) \end{aligned}$$

Again, from (3.1) and (3.2), we obtain

$$\begin{aligned} \|x_{n+1} - q\|^2 &\le \alpha_n\|u - q\|^2 + (1-\alpha_n)\|z_n - q\|^2 \\ &\le \alpha_n\|u - q\|^2 + (1-\alpha_n)(H(Ty_n, q))^2 \\ &\le \alpha_n\|u - q\|^2 + (1-\alpha_n)\|y_n - q\|^2 \\ &\le \max\{\|u - q\|^2, \|x_n - q\|^2\}. \end{aligned}$$

By induction, we obtain from the last inequality that

$$\|x_{n+1} - q\|^2 \le \max\{\|u - q\|^2, \|x_1 - q\|^2\}, \forall n \in N.$$

Clearly, the sequence $\{\|x_n - q\|\}_{n\in N}$ is bounded and so is $\{x_n\}_{n\in N}$. Consquently, using (3.1), the following sequences $\{y_n\}_{n\in N}$, $\{Ty_n\}_{n\in N}$, $\{Tx_n\}_{n\in N}$, $\{Sx_n\}_{n\in N}$ are bounded. Let

$$M = \sup\{\|u - q\|^2 - \|x_n - q\|^2 + \beta_n(1-\beta_n)g(\|v_n - w_n\|) : n \in N\} \ge 0. \qquad (3.4)$$

Then, it follows from (3.4) that

$$\beta_n(1-\beta_n)g(\|v_n - w_n\|) \le \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \alpha_n M. \qquad (3.5)$$

Next, we show that

$$\|x_{n+1} - q\|^2 \le (1-\alpha_n)\|x_n - q\|^2 + 2\alpha_n\langle u - q, J(x_{n+1} - q)\rangle. \qquad (3.6)$$

From Lemma 2.3, (3.1) and (3.3), we get

$$\begin{aligned} \|x_{n+1} - q\|^2 &\le (1-\alpha_n)^2\|z_n - q\|^2 + 2\alpha_n\langle u - q, J(x_{n+1} - q)\rangle \\ &\le (1-\alpha_n)^2(H(Ty_n, q))^2 + 2\alpha_n\langle u - q, J(x_{n+1} - q)\rangle \\ &\le (1-\alpha_n)^2\|y_n - q\|^2 + 2\alpha_n\langle u - q, J(x_{n+1} - q)\rangle \\ &\le (1-\alpha_n)\|x_n - q\|^2 + 2\alpha_n\langle u - q, J(x_{n+1} - q)\rangle. \qquad (3.7) \end{aligned}$$

Now, we show that $x_n \to q$ as $n \to \infty$. To do this, we consider two possible cases.

Case A. Suppose that $\{\|x_n - q\|\}_{n\in N}$ is a monotonically decreasing sequence, then there exists $n_0 \in N$ such that $\|x_n - q\|^2 - \|x_{n+1} - q\|^2 \to 0$ as $n \to \infty$. Thus, from (3.5) and conditions [(i) and (iii)], we have

$$\lim_{n\to\infty} g(\|v_n - w_n\|) = 0 \qquad (3.8)$$

and from the properties of g, we get

$$\lim_{n\to\infty} \|v_n - w_n\| = 0. \qquad (3.9)$$

Since $\{x_n\}_{n\in N}$ is bounded, there exists a subsequence $\{x_{n_k}\}_{k\in N}$ of $\{x_n\}_{n\in N}$ such that $x_{n_k} \rightharpoonup p \in D$ as $k \to \infty$. Furthermore, from (3.9), we have

$$\lim_{k\to\infty} \|v_{n_k} - w_{n_k}\| = 0. \tag{3.10}$$

Again, since (S, T) is assumed to satisfy jointly demiclosedness principle, it follows from (3.9) that $p \in \mathcal{F}$. Now, assume that there exists another subsequence $\{x_{n_i}\}_{k\in N}$ of $\{x_n\}_{n\in N}$ such that $x_{n_i} \rightharpoonup q \in D$ as $i \to \infty$ with $p \neq q$, where $q \in \mathcal{F}$. Then, (3.7) implies

$$\|p - q\|^2 \leq (1 - \alpha_n)\|p - q\|^2 + 2\alpha_n\langle u - q, J(p - q)\rangle. \tag{3.11}$$

Similarly,

$$\|q - p\|^2 \leq (1 - \alpha_n)\|q - p\|^2 + 2\alpha_n\langle u - q, J(q - p)\rangle. \tag{3.12}$$

Adding (3.11) and (3.12), we obtain $\|p - q\| \leq \|q - p\|$, which is a contradiction. Hence, $p = q$. Using the above information and Lemma 2.5, we get

$$\begin{aligned} \limsup_{n\to\infty}\langle u - q, J(x_{n+1} - q)\rangle &= \lim_{k\to\infty}\langle u - q, J(x_{n_k+1} - q)\rangle \\ &= \langle u - q, J(p - q)\rangle \leq 0. \end{aligned} \tag{3.13}$$

Hence, putting $\delta_n = \langle u - q, J(x_{n+1} - q)\rangle, \alpha_n = \gamma_n, s_n = \|x_n - q\|^2$, then it follows from (3.7), (3.13) and Lemma 2.1 that $\lim_{n\to} \|x_n - q\| = 0$, which is the desired result.

Case B. If $\{\|x_n - q\|\}_{n\in N}$ is not monotonically decreasing sequence, then there exists a nondecreasing sequence $\{n_i\}_{i\in N}$ of $\{n\}_{n\in N}$ such that

$$\|x_{n_i} - q\| < \|x_{n_i+1} - q\|. \tag{3.14}$$

Thus, by Lemma 2.4, there exists a nondecreasing sequence $\{x_{n_j}\}_{j\in N}$ such that

$$m_j \to \infty, \|x_{n_j} - q\| \leq \|x_{m_j+1} - q\| \quad \text{and} \quad \|x_j - q\| \leq \|x_{m_j+1} - q\| \tag{3.15}$$

By substituting m_j for n in (3.5), using the first part of the last inequality, we get

$$\begin{aligned} \beta_{m_j}(1 - \beta_{m_j})g(\|v_{m_j} - w_{m_j}\|) &\leq \|x_{m_j} - q\|^2 - \|x_{m_j+1} - q\|^2 + \alpha_{m_j}M \\ &\leq \alpha_{m_j}M, \forall j \in N. \end{aligned} \tag{3.16}$$

Thus, by conditions [(i) and (iii)]and the properties of g, we get

$$\lim_{j\to\infty} \|v_{m_j} - w_{m_j}\| = 0. \tag{3.17}$$

Using similar argument as in Case A, it is easy to show that

$$\limsup_{n\to\infty}\langle u - q, J(x_{n+1} - q)\rangle = \lim_{j\to\infty}\langle u - q, J(x_{m_j+1} - q)\rangle \leq 0. \tag{3.18}$$

Again, substituting m_j for n in (3.7), we have

$$\|x_{m_j+1} - q\|^2 \leq (1 - \alpha_{m_j})\|x_{m_j} - q\|^2 + 2\alpha_{m_j}\langle u - q, J(x_{m_j+1} - q)\rangle. \tag{3.19}$$

Using the last inequality with $\alpha_{m_j} \in (0, 1)$, we obtain

$$0 \leq \|x_{m_j} - q\|^2 - |x_{m_j} - q\|^2 \leq 2\alpha_{m_j}[\langle u - q, J(x_{m_j+1} - q)\rangle - \|x_{m_j} - q\|].$$

Hence, (3.18) yields

$$\lim_{j\to\infty} \|x_{m_j} - q\| = 0. \tag{3.20}$$

Also, from (3.19) and (3.20), we have

$$\lim_{j\to\infty} \|x_{m_j+1} - q\| = 0. \tag{3.21}$$

Finally, from (3.21) and the second part of the inequalities in (3.15), for all $j \in N$, we have $x_j \to q$ as $j \to \infty$.Thus, we have $x_n \to q$ as $n \to \infty$ as desired. This completes the proof. This completes the proof. □

Theorem 3.3. *Let E be a Banach space and D a nonempty, closed and convex subset of E and $v \in D$. Let $S, T : D \longrightarrow \mathcal{P}(D)$ be two multivalued type one quasi-nonexpansive multivalued self mappings from D into the family of proximinal subsets of D such that $\mathcal{F} = F(S) \cap F(T) \neq \emptyset$ is closed and convex.. Let S, T satifies jointly demiclosedness principle on D and $\{x_n\}$ be the sequence defined by*

$$\begin{cases} x_1 = x \in D; \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n; \\ y_n = (1-\beta_n)v_n + \beta_n w_n, \forall n \in N \end{cases} \tag{3.22}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$, $v_n \in Sx_n$ with $\|x_n - v_n\| = d(x_n, Sx_n)$ and $w_n \in Tx_n$ with $\|x_n - w_n\| = d(x_n, Tx_n)$. If the following conditions hold:

i. $\lim_{n\to\infty} \alpha_n = 0$
ii. $\sum_{n=1}^{\infty} \alpha_n = \infty$;
iii. $0 < \liminf_{n\to\infty} \beta_n(1-\beta_n) \leq \limsup_{n\to\infty} \beta_n(1-\beta_n) < 1$.

Then, the sequence converges strongly to $Q_{\mathcal{F}}v$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retaction from E onto $\mathcal{F}$.

Proof. Since S and T are multivalued quasi-nonexpansive mappings, it follows that $\mathcal{F}$ is closed and convex. Set $z = Q_{\mathcal{F}}v$. Let $q \in \mathcal{F}$ be fixed. Using Lemma 2.2, we can find a strictly increasing function $g : [0,\infty) \longrightarrow [0,\infty)$ with $g(0) = 0$ such that the following estimates remain valid:

$$\begin{aligned} \|y_n - q\|^2 &\leq (1-\beta_n)\|v_n - q\|^2 + \beta_n\|w_n - q\|^2 - \beta_n(1-\beta_n)g(\|v_n - u_v\|) \\ &\leq (1-\beta_n)\|x_n - q\|^2 + \beta_n\|x_n - q\|^2 - \beta_n(1-\beta_n)g(\|v_n - u_v\|) \\ &= \|x_n - q\|^2 - \beta_n(1-\beta_n)g(\|v_n - u_v\|) \qquad (3.23) \\ &\leq \|x_n - q\|^2. \qquad (3.24) \end{aligned}$$

Again, from (3.22) and (3.23), we obtain

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \alpha_n\|u - q\|^2 + (1-\alpha_n)\|y_n - q\|^2 \\ &\leq \alpha_n\|u - q\|^2 + (1-\alpha_n)\|x_n - q\|^2 \\ &\leq \max\{\|u - q\|^2, \|x_n - q\|^2\}. \end{aligned}$$

By induction, we obtain from the last inequality that

$$\|x_{n+1} - q\|^2 \leq \max\{\|u - q\|^2, \|x_1 - q\|^2\}, \forall n \in N.$$

Clearly, the sequence $\{\|x_n - q\|\}_{n\in N}$ is bounded and so is $\{x_n\}_{n\in N}$. Consquently, using (3.22), the following sequences $\{y_n\}_{n\in N}$, $\{Ty_n\}_{n\in N}$, $\{Tx_n\}_{n\in N}$, $\{Sx_n\}_{n\in N}$ are bounded. Let

$$M = \sup\{\|u-q\|^2 - \|x_n - q\|^2 + \beta_n(1-\beta_n)g(\|v_n - w_n\|) : n \in N\} \geq 0. \quad (3.25)$$

Then, it follows from (3.25) that

$$\beta_n(1-\beta_n)g(\|v_n - w_n\|) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \alpha_n M. \quad (3.26)$$

Next, we show that

$$\|x_{n+1} - q\|^2 \leq (1-\alpha_n)\|x_n - q\|^2 + 2\alpha_n\langle u - q, J(x_{n+1} - q)\rangle. \quad (3.27)$$

From Lemma 2.3, (3.22) and (3.24), we get

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq (1-\alpha_n)^2\|z_n - q\|^2 + 2\alpha_n\langle u-q, J(x_{n+1}-q)\rangle \\ &\leq (1-\alpha_n)^2(H(Ty_n, q))^2 + 2\alpha_n\langle u-q, J(x_{n+1}-q)\rangle \\ &\leq (1-\alpha_n)^2\|y_n - q\|^2 + 2\alpha_n\langle u-q, J(x_{n+1}-q)\rangle \\ &\leq (1-\alpha_n)\|x_n - q\|^2 + 2\alpha_n\langle u-q, J(x_{n+1}-q)\rangle. \end{aligned} \quad (3.28)$$

Now, we show that $x_n \to q$ as $n \to \infty$. To do this, we consider two possible cases.

Case A. Suppose $\{\|x_n - q\|\}_{n\in N}$ is a monotonically decreasing sequence, then there exists $n_0 \in N$ such that $\|x_n - q\|^2 - \|x_{n+1} - q\|^2 \to 0$ as $n \to \infty$. Thus, from (3.26) and conditions [(i) and (iii)], we have

$$\lim_{n\to\infty} g(\|v_n - w_n\|) = 0 \quad (3.29)$$

and from the properties of g, we get

$$\lim_{n\to\infty} \|v_n - w_n\| = 0. \quad (3.30)$$

Since $\{x_n\}_{n\in N}$ is bounded, there exists a subsequence $\{x_{n_k}\}_{k\in N}$ of $\{x_n\}_{n\in N}$ such that $x_{n_k} \rightharpoonup p \in D$ as $k \to \infty$. Furthermore, from (3.9), we have

$$\lim_{k\to\infty} \|v_{n_k} - w_{n_k}\| = 0. \quad (3.31)$$

Again, since (S, T) is assumed to satisfy jointly demiclosedness principle, it follows from (3.30) that $p \in \mathcal{F}$. Now, assume that there exists another subsequence $\{x_{n_i}\}_{k\in N}$ of $\{x_n\}_{n\in N}$ such that $x_{n_i} \rightharpoonup q \in D$ as $i \to \infty$ with $p \neq q$, where $q \in \mathcal{F}$. Then, (3.28) implies

$$\|p - q\|^2 \leq (1-\alpha_n)\|p - q\|^2 + 2\alpha_n\langle u - q, J(p - q)\rangle. \quad (3.32)$$

Similarly,

$$\|q - p\|^2 \leq (1-\alpha_n)\|q - p\|^2 + 2\alpha_n\langle u - q, J(q - p)\rangle. \quad (3.33)$$

Adding (3.32) and (3.33), we obtain $\|p - q\| \leq \|q - p\|$, which is a contradiction. Hence, $p = q$. Using the above information and Lemma 2.5, we get

$$\begin{aligned} \limsup_{n\to\infty}\langle u - q, J(x_{n+1} - q)\rangle &= \lim_{k\to\infty}\langle u - q, J(x_{n_k+1} - q)\rangle \\ &= \langle u - q, J(p - q)\rangle. \\ &\leq 0 \end{aligned} \quad (3.34)$$

Hence, putting $\delta_n = \langle u-q, J(x_{n+1}-q)\rangle, \alpha_n = \gamma_n, s_n = \|x_n - q\|^2$, then it follows from (3.28), (3.34) and Lemma 2.1 that $\lim_{n\to} \|x_n - q\| = 0$, which is the desired result.

Case B. If $\{\|x_n - q\|\}_{n\in N}$ is not monotonically decreasing sequence, then there exists a nondecreasing sequence $\{n_i\}_{i\in N}$ of $\{n\}_{n\in N}$ such that

$$\|x_{n_i} - q\| < \|x_{n_i+1} - q\|. \tag{3.35}$$

Thus, by Lemma 2.4, there exists a nondecreasing sequence $\{x_{n_j}\}_{j\in N}$ such that

$$m_j \to \infty, \|x_{n_j} - q\| \leq \|x_{m_j+1} - q\| \quad \text{and} \quad \|x_j - q\| \leq \|x_{m_j+1} - q\| \tag{3.36}$$

By substituting m_j for n in (3.26), using the first part of the last inequality, we get

$$\begin{aligned} \beta_{m_j}(1-\beta_{m_j})g(\|v_{m_j} - w_{m_j}\|) &\leq \|x_{m_j} - q\|^2 - \|x_{m_j+1} - q\|^2 + \alpha_{m_j} M \\ &\leq \alpha_{m_j} M, \forall j \in N. \end{aligned} \tag{3.37}$$

Thus, by conditions [(i) and (iii)]and the properties of g, we get

$$\lim_{j\to\infty} \|v_{m_j} - w_{m_j}\| = 0. \tag{3.38}$$

Using similar argument as in Case A, it is easy to show that

$$\begin{aligned} \limsup_{n\to\infty} \langle u-q, J(x_{n+1}-q)\rangle &= \lim_{j\to\infty} \langle u-q, J(x_{m_j+1}-q)\rangle \\ &\leq 0. \end{aligned} \tag{3.39}$$

Again, substituting m_j for n in (3.28), we have

$$\|x_{m_j+1} - q\|^2 \leq (1-\alpha_{m_j})\|x_{m_j} - q\|^2 + 2\alpha_{m_j}\langle u-q, J(x_{m_j+1}-q)\rangle. \tag{3.40}$$

Using the last inequality with $\alpha_{m_j} \in (0,1)$, we obtain

$$0 \leq \|x_{m_j} - q\|^2 - |x_{m_j} - q\|^2 \leq 2\alpha_{m_j}[\langle u-q, J(x_{m_j+1}-q)\rangle - \|x_{m_j} - q\|].$$

Hence, from (3.39), we obtain

$$\lim_{j\to\infty} \|x_{m_j} - q\| = 0. \tag{3.41}$$

Also, from (3.40) and (3.41), we have

$$\lim_{j\to\infty} \|x_{m_j+1} - q\| = 0. \tag{3.42}$$

Finally, from (3.42) and the second part of the inequalities in (3.36), for all $j \in N$, we have $x_j \to q$ as $j \to \infty$.Thus, we have $x_n \to q$ as $n \to \infty$ as desired. This completes the proof. This completes the proof. □

Corollary 3.4. *Let E be a Banach space and D a nonempty, closed and convex subset of E and $v \in D$. Let $S : D \longrightarrow \mathcal{P}(D)$ be multivalued nonspreading-type mapping and $T : D \longrightarrow mathcal P(D)$ be multivalued nonexpansive mapping such that $\mathcal{F} = F(S) \cap F(T) \neq \emptyset$ is closed and convex. Let S, T satisfies jointly demiclosedness principle on D and $\{x_n\}$ be the sequence defined by*

$$\begin{cases} x_1 = x \in D; \\ x_{n+1} = \alpha_n u + (1-\alpha_n) z_n; \\ y_n = (1-\beta_n)v_n + \beta_n w_n, \forall n \in N, \end{cases} \tag{3.43}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences in* $(0,1)$, $v_n \in Sx_n$ *with* $\|x_n - v_n\| = d(x_n, Sx_n)$, $w_n \in Tx_n$ *with* $\|x_n - w_n\| = d(x_n, Tx_n)$ *and* $z_n \in Ty_n$ *with* $\|y_n - z_{n,i}\| = d(y_n, Ty_n)$. *Under the conditions of Theorem 3.1, the sequence defined by* (3.43) *converges strongly to* $Q_{\mathcal{F}}v$, *where* $Q_{\mathcal{F}}$ *is the sunny nonexpansive retaction from* E *onto* $\mathcal{F}$.

Corollary 3.5. *Let* E *be a Banach space and* D *a nonempty, closed and convex subset of* E *and* $v \in D$. *Let* $S : D \longrightarrow mathcal P(D)$ *be multivalued nonspreading-type mapping and* $T : D \longrightarrow mathcal P(D)$ *be mutivalued nonexpansive mapping such that* $\mathcal{F} = F(S) \cap F(T) \neq \emptyset$ *is closed and convex.. Let* S, T *satisfies jointly demiclosedness principle on* D *and* $\{x_n\}$ *be the sequence defined by*

$$\begin{cases} x_1 = x \in D; \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n; \\ y_n = (1-\beta_n)v_n + \beta_n w_n, \forall n \in N, \end{cases} \tag{3.44}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences in* $(0,1)$, $v_n \in Sx_n$ *with* $\|x_n - v_n\| = d(x_n, Sx_n)$ *and* $w_n \in Tx_n$ *with* $\|x_n - w_n\| = d(x_n, Tx_n)$. *If the conditions of Theorem 3.1 hold, then the sequence defined by* (3.44) *converges strongly to* $Q_{\mathcal{F}}v$, *where* $Q_{\mathcal{F}}$ *is the sunny nonexpansive retaction from* E *onto* $\mathcal{F}$.

Remark 3.6. (i) We proposed two iteration schemes and established strong convergence results of the schemes to the common fixed points of two multivalued quasi-nonexpansive mappings satisfying the jointly demiclosedness principle. (ii) Corollary 3.3 and Corollary 3.4 provided affirmative answers to the open question 1.1 raised in [1] in a more general setting. (iii) We neither assumed the compactness condition on the space and the operators nor imposed any strict condition on the set of fixed points of the operators in obtaining our results.

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