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EXISTENCE RESULTS FOR FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING TWO CAPUTO DERIVATIVES WITH NONLOCAL CONDITIONS

MOHAMED HOUAS

Abstract. In this paper, we study existence and uniqueness of solutions for a nonlinear fractional differential equations involving two Caputo fractional derivatives with nonlocal and integral boundary conditions. We use Banach's contraction principle to prove an existence-uniqueness result. Then, by using the O'Regan fixed point theorem, we prove an existence result. Finally, illustrative examples of our main results are presented.

1. Introduction

The fractional differential equations theory arises in many engineering and scientific disciplines such as mechanics, physics, chemistry, biology, control theory and signal processing. For more details, we refer to [7,8,14,15]. Many studies on fractional differential equations, involving different operators such as Riemann-Liouville operators, [18,22], Caputo operators [6,9,10,14,23] and Hadamard operators [1,21] appeared during the past three decades. The existence and uniqueness of solutions for nonlinear fractional differential equations have been established and studied [2,3,11–13]. Recently, much more attention has been focused on the study of the existence of solutions for boundary value problems of fractional differential equations with nonlocal boundary conditions by the use of techniques of nonlinear analysis such as Banach's contraction principle, fixed-point theorems, Leray-Schauder theory, etc. [3,10,19,22,23] and the references cited therein.

In this work, we discuss the existence and uniqueness of solutions for nonlinear fractional differential equations involving two Caputo fractional derivatives with

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nonlocal and integral boundary conditions of the form.

$$\begin{cases} D^{\beta}\left(D^{\alpha}+\lambda\right)x\left(t\right)=f\left(t,x\left(t\right)\right)+\sum\limits_{j=1}^{m}I^{\sigma_j}g_j\left(t,x\left(t\right)\right),\ t\in J:=\left[0,T\right], \\ \\ x\left(0\right)=x_0+h\left(x\right),\ I^{p}x\left(T\right)=\sum\limits_{i=1}^{k}\mu_i I^{p}x\left(\eta_i\right), \end{cases} \tag{1.1}$$

where D^{β}, D^{α} denote the Caputo fractional derivatives, $0<\alpha,\beta\leq 1, 1<\alpha+\beta\leq 2$ and I^{p}, I^{σ_j} are the Riemann-Liouville fractional integral, with $0<p,\sigma_j<1, f,g_j : J\times\mathbb{R}\to\mathbb{R}\ (j=1,...,m), h : C\left(J,\mathbb{R}\right)\to\mathbb{R}$ are given continuous functions, $x_0,\lambda,\mu_i\in\mathbb{R}, 0<\eta_i<T, i=1,...,k, k\geq 2$ such that $\sum\limits_{i=1}^{k}\mu_i\eta_i^{\alpha+p}\neq T^{\alpha+p}$. The nonlocal condition involves $h\left(x\right)=\sum\limits_{l=1}^{q}c_l x\left(t_l\right)$, where $c_l, l=1,2,...,q,$ are given constants and $0<t_1<t_2<....<t_l\leq T$, can be more useful than the standard initial condition to describe some physical phenomena. For more details we refer to the work by Byszewski [4,5].

The rest of the paper is organized as follows. In Section 2 we recall some preliminary facts that we need in the sequel. In Section 3 we present our existence and uniqueness results. Examples illustrating the obtained results are presented in Section 4.

2. Preliminaries

In this section, we present some useful definitions and lemmas [17]:

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha>0$, for a continuous function f on $[a,b]$ is defined as:

$$I^{\alpha}f\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_a^t\left(t-\tau\right)^{\alpha-1}f\left(\tau\right)d\tau,\ a\leq t\leq b,$$

where $\Gamma\left(\alpha\right):=\int_0^{\infty}e^{-u}u^{\alpha-1}du$.

Definition 2.2. The fractional derivative of $f\in C^{n}\left(\left[a,b\right]\right)$ in the Caputo's sense is defined as:

$$D^{\alpha}f\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\int_a^t\left(t-\tau\right)^{n-\alpha-1}f^{(n)}\left(\tau\right)d\tau,\ n-1<\alpha,\ n\in N^{*},\ a\leq t\leq b.$$

The following lemma give some properties of Riemann-Liouville fractional integrals and Caputo fractional derivative [14,16]:

Lemma 2.3. *Let $r,s>0, s>r,\ f\in L_1([a,b])$. Then*

$$I^{r}I^{s}f(t)=I^{r+s}f(t),\ D^{s}I^{s}f(t)=f(t)\ \textit{and}\ D^{r}I^{s}f(t)=I^{s-r}f(t), t\in\left[a,b\right].$$

We also give the following lemmas [14,16]:

Lemma 2.4. *For $\alpha > 0$, the general solution of the fractional differential equation $D^{\alpha}x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1,\ n = [\alpha] + 1$.

Lemma 2.5. *Let $\alpha > 0$. Then*

$$I^{\alpha}D^{\alpha}x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1},$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, n = [\alpha] + 1$.

We need also the following auxiliary result:

Lemma 2.6. *Let $\sum_{i=1}^{k}\mu_i\eta_i^{\alpha+p} \neq T^{\alpha+p}$. For any $\varphi \in C(J, \mathbb{R})$, the unique solution of the fractional boundary value problem*

$$\begin{cases} D^{\beta}(D^{\alpha}+\lambda)x(t) = \varphi(t),\ t \in J,\ 0 < \alpha,\ \beta \leq 1, 0 < \alpha+\beta \leq 2, \\ x(0) = x_0 + h(x),\ I^p x(T) = \sum_{i=1}^{k}\mu_i I^p x(\eta_i),\ 0 < \eta_i < T, \end{cases} \tag{2.1}$$

is given by:

$$\begin{aligned} & x(t) \\ = & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\varphi(s)ds - \lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds \\ & -\frac{\Delta t^{\alpha}}{\Gamma(\alpha+1)}\left[\sum_{i=1}^{k}\mu_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)}\varphi(s)ds - \lambda\sum_{i=1}^{k}\mu_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha+p-1}}{\Gamma(\alpha+p)}x(s)ds \right. \\ & \left. -\int_0^T\frac{(T-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)}\varphi(s)ds + \lambda\int_0^T\frac{(T-s)^{\alpha+p-1}}{\Gamma(\alpha+p)}x(s)ds\right] \\ & +\left[\frac{\Delta t^{\alpha}}{\Gamma(\alpha+1)\Gamma(p+1)}\left(\sum_{i=1}^{k}\mu_i\eta_i^{p} - T^{p}\right)+1\right](x_0+h(x)), \end{aligned} \tag{2.2}$$

where

$$\Delta := \frac{\Gamma(\alpha+p+1)}{\sum_{i=1}^{k}\mu_i\eta_i^{\alpha+p} - T^{\alpha+p}},\quad \sum_{i=1}^{k}\mu_i\eta_i^{\alpha+p} \neq T^{\alpha+p}. \tag{2.3}$$

Proof. For $0 < \alpha, \beta \leq 1, 0 < \alpha+\beta \leq 2$, by Lemma 2.3 and Lemma 2.4, we know that the general solution of the differential equation $D^{\beta}(D^{\alpha}+\lambda)x(t) = \varphi(t)$ can be written as

$$x(t) = I^{\alpha+\beta}\varphi(t) - \lambda I^{\alpha}x(t) - c_0\frac{t^{\alpha}}{\Gamma(\alpha+1)} - c_1, \tag{2.4}$$

where c_0 and c_1 are arbitrary constants. By taking the Riemann-Liouville fractional integral of order p, we get

$$I^p x(t) = I^{p+\alpha+\beta}\varphi(t) - \lambda I^{p+\alpha}x(t) - c_0\frac{t^{p+\alpha}}{\Gamma(p+\alpha+1)} - \frac{t^p}{\Gamma(p+1)}c_1. \tag{2.5}$$

Using the boundary conditions, we obtain

$$\begin{aligned} c_1 &= -\left(x_0 + h\left(x\right)\right), \\ c_0 &= \Delta\left[\sum_{i=1}^{k}\mu_i I^{\alpha+\beta+p}\varphi(\eta_i) - \lambda\sum_{i=1}^{k}\mu_i I^{\alpha+p}x(\eta_i) - I^{\alpha+\beta+p}\varphi(T)\right. \\ &\quad \left. +\lambda I^{\alpha+p}x(T) - \frac{1}{\Gamma(p+1)}\left(\sum_{i=1}^{k}\mu_i\eta_i^p - T^p\right)(x_0 + h(x))\right], \end{aligned}$$

where Δ is defined by (2.3). Substituting the value of c_0 and c_1 in (2.4), we obtain (2.2). □

3. Main Results

We denote by $X = C(J,\mathbb{R})$ the Banach space of all continuous functions from J into $\mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\left\{|x\left(t\right)| : t \in J\right\}$.

In view of Lemma 2.5 we define an operator: $\phi : X \to X$ as

$$\begin{aligned} &\phi x(t) \qquad (3.1) \\ = &\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s, x\left(s\right))ds + \sum_{j=1}^{m}\int_0^t \frac{(t-s)^{\alpha+\beta+\sigma_j-1}}{\Gamma(\alpha+\beta+\sigma_j)} g_j(s, x\left(s\right))ds \\ &-\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \\ &-\frac{\Delta t^\alpha}{\Gamma(\alpha+1)}\left[\sum_{i=1}^{k}\mu_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)} f(s, x\left(s\right))ds\right. \\ &+\sum_{j=1}^{m}\sum_{i=1}^{k}\mu_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha+\beta+p+\sigma_j-1}}{\Gamma(\alpha+\beta+p+\sigma_j)} g_j(s, x\left(s\right))ds \\ &-\lambda\sum_{i=1}^{k}\mu_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha+p-1}}{\Gamma(\alpha+p)} x(s)ds \\ &-\int_0^T \frac{(T-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)} f(s, x\left(s\right))ds - \sum_{j=1}^{m}\int_0^T \frac{(T-s)^{\alpha+\beta+p+\sigma_j-1}}{\Gamma(\alpha+\beta+p+\sigma_j)} g_j(s, x\left(s\right))ds \\ &\left.+\lambda\int_0^T \frac{(T-s)^{\alpha+p-1}}{\Gamma(\alpha+p)} x(s)ds\right] \\ &+\left[\frac{\Delta t^\alpha}{\Gamma(\alpha+1)\Gamma(p+1)}\left(\sum_{i=1}^{k}\mu_i\eta_i^p - T^p\right) + 1\right](x_0 + h(x)), \end{aligned}$$

Observe that problem (1.1) has solutions if and only if the operator equation $\phi x = x$. We put $\phi(x)(t) := \phi_1(x)(t) + \phi_2(x)(t)$, $t \in J$, where

$$\begin{aligned} &\phi_1 x(t) &(3.2)\\ = &\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s,x(s))ds + \sum_{j=1}^m \int_0^t \frac{(t-s)^{\alpha+\beta+\sigma_j-1}}{\Gamma(\alpha+\beta+\sigma_j)} g_j(s,x(s))ds \\ &-\lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds - \frac{\Delta t^\alpha}{\Gamma(\alpha+1)} \Bigg[\sum_{i=1}^k \mu_i \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)} f(s,x(s))ds \\ &+\sum_{j=1}^m \sum_{i=1}^k \mu_i \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha+\beta+p+\sigma_j-1}}{\Gamma(\alpha+\beta+p+\sigma_j)} g_j(s,x(s))ds \\ &-\lambda \sum_{i=1}^k \mu_i \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha+p-1}}{\Gamma(\alpha+p)} x(s)ds \\ &-\int_0^T \frac{(T-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)} f(s,x(s))ds \\ &-\sum_{j=1}^m \int_0^T \frac{(T-s)^{\alpha+\beta+p+\sigma_j-1}}{\Gamma(\alpha+\beta+p+\sigma_j)} g_j(s,x(s))ds + \lambda \int_0^T \frac{(T-s)^{\alpha+p-1}}{\Gamma(\alpha+p)} x(s)ds \Bigg], \end{aligned}$$

and

$$\phi x_2(t) = \left[\frac{\Delta t^\alpha}{\Gamma(\alpha+1)\Gamma(p+1)} \left(\sum_{i=1}^k \mu_i \eta_i^p - T^p \right) + 1 \right] (x_0 + h(x)). \tag{3.3}$$

For computational convenience, we set the notations:

$$\begin{aligned} \Lambda_1 \ : \ &= \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \sum_{j=1}^m \frac{T^{\alpha+\beta+\sigma_j}}{\Gamma(\alpha+\beta+\sigma_j+1)} &(3.4)\\ &+\frac{|\Delta| T^\alpha}{\Gamma(\alpha+1)} \Bigg(\sum_{i=1}^k \frac{|\mu_i| \eta_i^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)} + \sum_{j=1}^m \sum_{i=1}^k \frac{|\mu_i| \eta_i^{\alpha+\beta+p+\sigma_j}}{\Gamma(\alpha+\beta+p+\sigma_j+1)} \\ &+\frac{T^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)} + \sum_{j=1}^m \frac{T^{\alpha+\beta+p+\sigma_j}}{\Gamma(\alpha+\beta+p+\sigma_j+1)} \Bigg), \end{aligned}$$

$$\Lambda_2 := |\lambda| \left[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{|\Delta| T^\alpha}{\Gamma(\alpha+1)} \left(\sum_{i=1}^k \frac{|\mu_i| \eta_i^{\alpha+p}}{\Gamma(\alpha+p+1)} + \frac{T^{\alpha+p}}{\Gamma(\alpha+p+1)} \right) \right], \tag{3.5}$$

and

$$\Lambda_3 := \frac{|\Delta| T^\alpha}{\Gamma(\alpha+1)\Gamma(p+1)} \left| \sum_{i=1}^k \mu_i \eta_i^p - T^p \right| + 1. \tag{3.6}$$

Theorem 3.1. *Let $f, g_j : J \times \mathbb{R} \to \mathbb{R}, j = 1, ..., m$ be continuous functions. Assume that:*

(A_1) $|f(t,x) - f(t,y)| \leq N_1 |x-y|$, $|g_j(t,x(t)) - g_j(t,y(t))| \leq N_{j+1} |x-y|$, *for* $j = 1,...,m$, $t \in J, (x,y) \in \mathbb{R}^2, N_j > 0, j = 1,...,m+1$;

(A_2) *There exists a positive constant* $M < \frac{1-\Lambda_2}{\Lambda_3}$ *and a continuous function* $\theta : [0,\infty) \to [0,\infty)$ *such that* $\theta(v) \leq Mv$ *and* $|h(x) - h(y)| \leq \theta(\|x-y\|)$ *for all* $x, y \in C(J,\mathbb{R})$;

(A_3) $\Lambda_1 N + \Lambda_3 M < 1 - \Lambda_2$, *where* $N = \max\{N_j : j = 1,...,m+1\}$ *and* Λ_1, Λ_2 *and* Λ_3 *are defined by (3.4), (3.5) and (3.6) respectively.*

Then the fractional boundary value problem (1.1) has a unique solution.

Proof. For $x, y \in C(J,\mathbb{R})$ and for each $t \in J$, from the definition of ϕ and assumptions and (A_1) and (A_2) we have:

$$\begin{aligned}
&\|\phi x - \phi y\| \qquad (3.7)\\
\leq\ & \sup_{t\in[0,T]} \left\{ \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |f(s,x(s)) - f(s,y(s))| \, ds \right.\\
&+ \sum_{j=1}^m \int_0^t \frac{(t-s)^{\alpha+\beta+\sigma_j-1}}{\Gamma(\alpha+\beta+\sigma_j)} |g_j(s,x(s)) - g_j(s,y(s))| \, ds\\
&+ |\lambda| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s) - y(s)| \, ds\\
&+ \frac{|\Delta| t^\alpha}{\Gamma(\alpha+1)} \left[\sum_{i=1}^k |\mu_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)} |f(s,x(s)) - f(s,y(s))| \, ds \right.\\
&+ \sum_{j=1}^m \sum_{i=1}^k |\mu_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha+\beta+p+\sigma_j-1}}{\Gamma(\alpha+\beta+p+\sigma_j)} |g_j(s,x(s)) - g_j(s,y(s))| \, ds\\
&+ |\lambda| \sum_{i=1}^k |\mu_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha+p-1}}{\Gamma(\alpha+p)} |x(s) - y(s)| \, ds\\
&+ \int_0^T \frac{(T-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)} |f(s,x(s)) - f(s,y(s))| \, ds\\
&+ \sum_{j=1}^m \int_0^T \frac{(T-s)^{\alpha+\beta+p+\sigma_j-1}}{\Gamma(\alpha+\beta+p+\sigma_j)} |g_j(s,x(s)) - g_j(s,y(s))| \, ds\\
&\left. + \lambda \int_0^T \frac{(T-s)^{\alpha+p-1}}{\Gamma(\alpha+p)} |x(s) - y(s)| \, ds \right]\\
&\left. + \left[\frac{|\Delta| t^\alpha}{\Gamma(\alpha+1)\Gamma(p+1)} \left| \sum_{i=1}^k \mu_i \eta_i^p - T^p \right| + 1 \right] |h(x) - h(y)| \right\},
\end{aligned}$$

$$
\begin{aligned}
&\leq \left[\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\sum_{j=1}^{m}\frac{T^{\alpha+\beta+\sigma_j}}{\Gamma(\alpha+\beta+\sigma_j+1)}\right.\\
&+\frac{|\Delta|\,T^{\alpha}}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{k}\frac{|\mu_i|\,\eta_i{}^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)}+\sum_{j=1}^{m}\sum_{i=1}^{k}\frac{|\mu_i|\,\eta_i{}^{\alpha+\beta+p+\sigma_j}}{\Gamma(\alpha+\beta+p+\sigma_j+1)}\right.\\
&\left.\left.+\frac{T^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)}+\sum_{j=1}^{m}\frac{T^{\alpha+\beta+p+\sigma_j}}{\Gamma(\alpha+\beta+p+\sigma_j+1)}\right)\right]N\,\|x-y\|\\
&+|\lambda|\left[\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{|\Delta|\,T^{\alpha}}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{k}\frac{|\mu_i|\,\eta_i^{\alpha+p}}{\Gamma(\alpha+p+1)}+|\lambda|\,\frac{T^{\alpha+p}}{\Gamma(\alpha+p+1)}\right)\right]\|x-y\|\\
&+\left[\frac{|\Delta|\,T^{\alpha}}{\Gamma(\alpha+1)\Gamma(p+1)}\left|\sum_{i=1}^{k}\mu_i\eta_i^{p}-T^{p}\right|+1\right]M\,\|x-y\|\\
&= (\Lambda_1N+\Lambda_2+\Lambda_3M)\,\|x-y\|\,,
\end{aligned}
$$

Thus

$$\|\phi(x)-\phi(y)\|\leq(\Lambda_1N+\Lambda_2+\Lambda_3M)\,\|x-y\|\,. \tag{3.8}$$

Thanks to (A_3), we conclude that ϕ is a contraction. As a consequence of Banach fixed point theorem, we deduce that ϕ has a fixed point which is a solution of the problem (1.1).

□

Next, we introduce a fixed point theorem due to O'Regan [20], which will be used to establish the next main result.

Lemma 3.2. *Denote by V an open set in a closed, convex set C of a Banach space E. Assume $0\in V$. Also assume that $\phi\left(\bar{V}\right)$ is bounded and that $\phi:\bar{V}\to C$ is given by $\phi=\phi_1+\phi_2$, in which $\phi_1:\bar{V}\to E$ is continuous and completely continuous and $\phi_2:\bar{V}\to E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\theta:[0,\infty)\to[0,\infty)$ satisfying $\theta(v)<v$ for $v>0$ such that $\|\phi_2(x)-\phi_2(y)\|\leq\theta(\|x-y\|)$ for all $x,y\in\bar{V}$). Then, either*

(I) ϕ has a fixed point $x\in\bar{V}$; or

(II) there exist a point $x\in\partial V$ and $0<\mu<1$ with $x=\mu\phi(x)$,

where $\bar{V}$, (respectively ∂V,) represents the closure, (respectively the boundary) of V.

Theorem 3.3. *Suppose that $f,g_j:J\times\mathbb{R}\to\mathbb{R}$, $j=1,...,m$, are continuous functions. Suppose that (A_1) and (A_2) hold. In addition we assume that:*

(A_4) $h(0)=0$;

(A_5) *there exist a nonnegative functions* $\gamma(t), \gamma_j(t) \in C(J, \mathbb{R}), (j = 1, ..., m)$ *and a nondecreasing functions* $\psi, \psi_j : [0, \infty) \to [0, \infty)$, $(j = 1, ..., m)$ *such that* $|f(t, x(t))| \le \gamma(t)\psi(|x|), |g_j(t, x(t))| \le \gamma_j(t)\psi_j(|x|)$;

(A_6) $\sup\limits_{\delta\in(0,\infty)} \frac{\delta}{\Pi\|\gamma\|\psi(\delta)+\sum\limits_{j=1}^{m}\Pi_j\|\gamma_j\|\psi_j(\delta)+\Lambda_3|x_0|} > \frac{1}{1-(\Lambda_2+\Lambda_3 M)}$, *where*

$$\Pi := \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\Delta|\, T^{\alpha}}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{k}\frac{|\mu_i|\, {\eta_i}^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)} + \frac{T^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)}\right), \tag{3.9}$$

$$\Pi_j := \frac{T^{\alpha+\beta+\sigma_j}}{\Gamma(\alpha+\beta+\sigma_j+1)} + \frac{|\Delta|\, T^{\alpha}}{\Gamma(\alpha+1)}\left[\sum_{i=1}^{k}\frac{|\mu_i|\, {\eta_i}^{\alpha+\beta+p+\sigma_j}}{\Gamma(\alpha+\beta+p+\sigma_j+1)}\right. \tag{3.10}$$

$$\left. + \frac{T^{\alpha+\beta+p+\sigma_j}}{\Gamma(\alpha+\beta+p+\sigma_j+1)}\right], j = 1, ...m,$$

and Λ_2, Λ_3 *are defined by (3.5) and (3.6) respectively.*

Then the fractional boundary value problem (1.1) has at least one solution on J.

Proof. Consider the operator $\phi : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ defined by:

$$\phi\,(x)\,(t) = \phi_1\,(x)\,(t) + \phi_2\,(x)\,(t)\,,\ t \in J, \tag{3.11}$$

where the operators ϕ_1 and ϕ_2 are defined respectively in (3.2) and (3.3).
From (A_6) there exist a number $\delta_0 > 0$ such that

$$\frac{\delta_0}{\Pi\,\|\gamma\|\,\psi(\delta_0) + \sum\limits_{j=1}^{m}\Pi_j\,\|\gamma_j\|\,\psi_j(\delta_0) + \Lambda_3\,|x_0|} > \frac{1}{1-(\Lambda_2+\Lambda_3 M)}. \tag{3.12}$$

We shall prove that the operators ϕ_1 and ϕ_2 satisfy all the conditions in Lemma 3.2.

*Step*1 : The operator ϕ_1 is continuous and completely continuous. Let us consider the set

$$\bar{\Omega}_{\delta_0} = \{x \in C(J, \mathbb{R}) : \|x\| \le \delta_0\}\,, \tag{3.13}$$

and show that $\phi_1\left(\bar{\Omega}_{\delta_0}\right)$ is bounded. Taking $x \in \bar{\Omega}_{\delta_0}$, then for each $t \in J$, we have:

$$
\begin{aligned}
&\|\phi_1 x\| \qquad (3.14)\\
\leq\; & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left|f(s,x(s))\right| ds+\sum_{j=1}^{m}\int_0^t \frac{(t-s)^{\alpha+\beta+\sigma_j-1}}{\Gamma(\alpha+\beta+\sigma_j)}\left|g_j(s,x(s))\right| ds\\
&+|\lambda|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)|\,ds\\
&+\frac{|\Delta|\,t^{\alpha}}{\Gamma(\alpha+1)}\left[\sum_{i=1}^{k}|\mu_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)}\left|f(s,x(s))\right| ds\right.\\
&+\sum_{j=1}^{m}\sum_{i=1}^{k}|\mu_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha+\beta+p+\sigma_j-1}}{\Gamma(\alpha+\beta+p+\sigma_j)}\left|g_j(s,x(s))\right| ds\\
&+|\lambda|\sum_{i=1}^{k}|\mu_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha+p-1}}{\Gamma(\alpha+p)}|x(s)|\,ds\\
&+\int_0^T\frac{(T-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)}\left|f(s,x(s))\right| ds\\
&+\sum_{j=1}^{m}\int_0^T\frac{(T-s)^{\alpha+\beta+p+\sigma_j-1}}{\Gamma(\alpha+\beta+p+\sigma_j)}\left|g_j(s,x(s))\right| ds\\
&\left.+|\lambda|\int_0^T\frac{(T-s)^{\alpha+p-1}}{\Gamma(\alpha+p)}|x(s)|\,ds\right],
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\|\phi_1 x\| \qquad (3.15)\\
\leq\; & \left[\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{|\Delta|\,T^{\alpha}}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{k}\frac{|\mu_i|\,\eta_i{}^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)}+\frac{T^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)}\right)\right]\\
&\|\gamma\|\,\psi(\sigma_0)+\sum_{j=1}^{m}\left(\frac{T^{\alpha+\beta+\sigma_j}}{\Gamma(\alpha+\beta+\sigma_j+1)}+\frac{|\Delta|\,T^{\alpha}}{\Gamma(\alpha+1)}\left[\sum_{i=1}^{k}\frac{|\mu_i|\,\eta_i{}^{\alpha+\beta+p+\sigma_j}}{\Gamma(\alpha+\beta+p+\sigma_j+1)}\right.\right.\\
&\left.\left.+\;\frac{T^{\alpha+\beta+p+\sigma_j}}{\Gamma(\alpha+\beta+p+\sigma_j+1)}\right]\right)\|\gamma_j\|\,\psi_j(\sigma_0)\\
&+|\lambda|\left[\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{|\Delta|\,T^{\alpha}}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{k}\frac{|\mu_i|\,\eta_i{}^{\alpha+p}}{\Gamma(\alpha+p+1)}+\frac{T^{\alpha+p}}{\Gamma(\alpha+p+1)}\right)\right]\sigma_0\\
&\leq \Pi\,\|\gamma\|\,\psi(\delta_0)+\sum_{j=1}^{m}\Pi_j\,\|\gamma_j\|\,\psi_j(\delta_0)+\Lambda_3\delta_0.
\end{aligned}
$$

This proves that $\phi_1\left(\bar{\Omega}_{\delta_0}\right)$ is uniformly bounded. For $x \in \bar{\Omega}_{\delta_0}$ and $t_1, t_2 \in J$, such that $t_2 < t_1$. We have:

$$
\begin{aligned}
& \left|\phi(x)(t_1) - \phi(x)(t_2)\right| \qquad (3.16)\\
\leq\; & \frac{1}{\Gamma(\alpha+\beta)} \int_0^{t_1} \left[(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}\right] |f(s,x(s))|\, ds \\
& + \frac{1}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} \left[(t_2-s)^{\alpha+\beta-1}\right] |f(s,x(s))|\, ds \\
& + \sum_{j=1}^{m} \int_0^{t_1} \frac{\left[(t_2-s)^{\alpha+\beta+\sigma_j-1} - (t_1-s)^{\alpha+\beta+\sigma_j-1}\right]}{\Gamma(\alpha+\beta+\sigma_j)} |g_j(s,x(s))|\, ds \\
& + \sum_{j=1}^{m} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha+\beta+\sigma_j-1}}{\Gamma(\alpha+\beta+\sigma_j)} |g_j(s,x(s))|\, ds \\
& + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^{t_2} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] |x(s)|\, ds \\
& + \frac{|\lambda|}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left[(t_2-s)^{\alpha-1}\right] |x(s)|\, ds \\
& + \frac{|\Delta|\left[t_2^{\alpha} - t_1^{\alpha}\right]}{\Gamma(\alpha+1)} \left[\sum_{i=1}^{k} |\mu_i| \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)} |f(s,x(s))|\, ds\right. \\
& + \sum_{j=1}^{m}\sum_{i=1}^{k} |\mu_i| \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha+\beta+p+\sigma_j-1}}{\Gamma(\alpha+\beta+p+\sigma_j)} |g_j(s,x(s))|\, ds \\
& + |\lambda| \sum_{i=1}^{k} |\mu_i| \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha+p-1}}{\Gamma(\alpha+p)} |x(s)|\, ds \\
& + \int_0^{T} \frac{(T-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)} |f(s,x(s))|\, ds \\
& + \sum_{j=1}^{m} \int_0^{T} \frac{(T-s)^{\alpha+\beta+p+\sigma_j-1}}{\Gamma(\alpha+\beta+p+\sigma_j)} |g_j(s,x(s))|\, ds \\
& \left. + |\lambda| \int_0^{T} \frac{(T-s)^{\alpha+p-1}}{\Gamma(\alpha+p)} |x(s)|\, ds\right],
\end{aligned}
$$

Thus

$$
\begin{aligned}
& \left|\phi(x)(t_1) - \phi(x)(t_2)\right| \qquad (3.17)\\
\leq\; & \frac{\|\gamma\|\,\psi(\delta_0)}{\Gamma(\alpha+\beta+1)} \left[{t_2}^{\alpha+\beta} - {t_1}^{\alpha+\beta}\right] + \sum_{j=1}^{m} \frac{\|\gamma_j\|\,\psi_j(\delta_0)}{\Gamma(\alpha+\beta+\sigma_j+1)} \left[{t_2}^{\alpha+\beta+\sigma_j} - {t_1}^{\alpha+\beta+\sigma_j}\right] \\
& + \frac{|\lambda|\,\delta_0}{\Gamma(\alpha+1)} \left[{t_2}^{\alpha} - {t_1}^{\alpha}\right] + \frac{|\Delta|\left[t_2^{\alpha} - t_1^{\alpha}\right]}{\Gamma(\alpha+1)} \left[\|\gamma\|\,\psi(\delta_0) \sum_{i=1}^{k} \frac{|\mu_i|\,{\eta_i}^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)}\right. \qquad (3.18)
\end{aligned}
$$

$$+\sum_{j=1}^{m}\sum_{i=1}^{k}\frac{|\mu_i|\,\eta_i{}^{\alpha+\beta+p+\sigma_j}\,\|\gamma_j\|\,\psi_j(\delta_0)}{\Gamma(\alpha+\beta+p+\sigma_j+1)}+|\lambda|\,\delta_0\sum_{i=1}^{k}\frac{|\mu_i|\,\eta_i{}^{\alpha+p}}{\Gamma(\alpha+p+1)}$$
$$\left.+\frac{T^{\alpha+\beta+p}\,\|\gamma\|\,\psi(\delta_0)}{\Gamma(\alpha+\beta+p+1)}+\sum_{j=1}^{m}\frac{T^{\alpha+\beta+p+\sigma_j}\,\|\gamma_j\|\,\psi_j(\delta_0)}{\Gamma(\alpha+\beta+p+\sigma_j+1)}+\frac{|\lambda|\,\delta_0 T^{\alpha+p}}{\Gamma(\alpha+p+1)}\right],$$

which is independent of x and tends to zero as $t_1 \to t_2$. Thus, ϕ_1 is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\phi_1\left(\bar{\Omega}_{\delta_0}\right)$ is a relatively compact set. Now let $x_n \subset \bar{\Omega}_{\delta_0}$ with $\|x_n - x\| \to 0$. Then the limit

$$\|x_n(t) - x(t)\| \to 0$$

uniformly on J. From the uniform continuity of $f(t,x(t))$ and $g_j(t,x(t))$ on the compact set $J\times[-\delta_0,\delta_0]$, it follows that

$$\|f(t,x_n(t)) - f(t,x(t))\| \to 0$$

and

$$\|g_j(t,x_n(t)) - g_j(t,x(t))\| \to 0$$

are uniformly valid on J. Hence

$$\|\phi_1(t,x_n(t)) - \phi_1(t,x(t))\| \to 0$$

as $n\to\infty$, which proves the continuity of ϕ_1. This completes the proof of *Step*1.

*Step*2 : The operator $\phi_2 : \bar{\Omega}_{\delta_0} \to C(J,\mathbb{R})$ is contractive. This is consequence of (A_2).

*Step*3 : The set $\phi_2\left(\bar{\Omega}_{\delta_0}\right)$ is bounded. For any $x\in\bar{\Omega}_{\delta_0}$and by (A_2) and (A_4), we obtain

$$\begin{aligned}\|\phi_2 x\| &\leq \left[\frac{|\Delta|\,T^{\alpha}}{\Gamma(\alpha+1)\Gamma(p+1)}\left(\sum_{i=1}^{k}\mu_i\eta_i^p - T^p\right)+1\right](|x_0|+M\delta_0)\\ &= \Lambda_3\,|x_0|+M\delta_0), \end{aligned}\tag{3.19}$$

combining, with the set $\phi_1\left(\bar{\Omega}_{\delta_0}\right)$ is bounded (*Step*1), we have the set $\phi\left(\bar{\Omega}_{\delta_0}\right)$ is bounded.

*Step*4 : Finally, it is to show that the case (II) in Lemma 3.2 does not occur. To this end, we perform the argument by contradiction. We suppose that (II) holds. Then, there exist $\rho\in(0,1)$ and $x\in\partial\Omega_{\delta_0}$ such that $x=\rho\phi(x)$. So we have $\|x\| = \delta_0$ and

$$\begin{aligned} & x(t) \qquad (3.20) \\ = \; & \rho\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s,x(s))ds + \rho\sum_{j=1}^{m}\int_0^t \frac{(t-s)^{\alpha+\beta+\sigma_j-1}}{\Gamma(\alpha+\beta+\sigma_j)} g_j(s,x(s))ds \\ & -\rho\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds - \frac{\rho\Delta t^{\alpha}}{\Gamma(\alpha+1)}\left[\sum_{i=1}^{k}\mu_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)}f(s,x(s))ds\right. \\ & +\sum_{j=1}^{m}\sum_{i=1}^{k}\mu_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha+\beta+p+\sigma_j-1}}{\Gamma(\alpha+\beta+p+\sigma_j)}g_j(s,x(s))ds - \lambda\sum_{i=1}^{k}\mu_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha+p-1}}{\Gamma(\alpha+p)}x(s)ds \\ & -\int_0^T\frac{(T-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)}f(s,x(s))ds - \sum_{j=1}^{m}\int_0^T\frac{(T-s)^{\alpha+\beta+p+\sigma_j-1}}{\Gamma(\alpha+\beta+p+\sigma_j)}g_j(s,x(s))ds \\ & \left.+\lambda\int_0^T\frac{(T-s)^{\alpha+p-1}}{\Gamma(\alpha+p)}x(s)ds\right] \\ & +\rho\left[\frac{\Delta t^{\alpha}}{\Gamma(\alpha+1)\Gamma(p+1)}\left(\sum_{i=1}^{k}\mu_i\eta_i^{p} - T^{p}\right)+1\right](x_0+h(x)), \end{aligned}$$

In view of the assumptions (A_2) and $(A_4)-(A_6)$, we have

$$\begin{aligned} & \delta_0 \qquad (3.21) \\ \leq \; & \left[\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\Delta|\,T^{\alpha}}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{k}\frac{|\mu_i|\,{\eta_i}^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)} + \frac{T^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)}\right)\right] \\ & \|\gamma\|\,\psi(\sigma_0) + \sum_{j=1}^{m}\left(\frac{T^{\alpha+\beta+\sigma_j}}{\Gamma(\alpha+\beta+\sigma_j+1)} + \frac{|\Delta|\,T^{\alpha}}{\Gamma(\alpha+1)}\left[\sum_{i=1}^{k}\frac{|\mu_i|\,{\eta_i}^{\alpha+\beta+p+\sigma_j}}{\Gamma(\alpha+\beta+p+\sigma_j+1)}\right.\right. \\ & \left.\left.+\;\frac{T^{\alpha+\beta+p+\sigma_j}}{\Gamma(\alpha+\beta+p+\sigma_j+1)}\right]\right)\|\gamma_j\|\,\psi_j(\sigma_0) \\ & +|\lambda|\left[\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{|\Delta|\,T^{\alpha}}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{k}\frac{|\mu_i|\,{\eta_i}^{\alpha+p}}{\Gamma(\alpha+p+1)} + \frac{T^{\alpha+p}}{\Gamma(\alpha+p+1)}\right)\right]\delta_0 \\ & +\left[\frac{|\Delta|\,T^{\alpha}}{\Gamma(\alpha+1)\Gamma(p+1)}\left|\sum_{i=1}^{k}\mu_i\eta_i^{p} - T^{p}\right|+1\right](|x_0|+M\delta_0), \end{aligned}$$

which implies

$$\delta_0 \leq \Pi\,\|\gamma\|\,\psi(\delta_0) + \sum_{j=1}^{m}\Pi_j\,\|\gamma_j\|\,\psi_j(\delta_0) + (\Lambda_2+\Lambda_3 M)\,\delta_0 + \Lambda_3\,|x_0|\,. \qquad (3.22)$$

Thus,

$$\frac{\delta_0}{\Pi\,\|\gamma\|\,\psi(\delta_0) + \sum\limits_{j=1}^{m}\Pi_j\,\|\gamma_j\|\,\psi_j(\delta_0) + \Lambda_3\,|x_0|} \leq \frac{1}{1-(\Lambda_2+\Lambda_3 M)}, \qquad (3.23)$$

which contradicts (3.12). Consequently, we have proved that the operators ϕ_1 and ϕ_2 satisfy all the conditions in Lemma 3.2. Hence, the operator ϕ has at least one fixed point $x \in \bar{\Omega}_{\delta_0}$, which is the solution of the fractional boundary value problem (1.1). □

4. Examples

To illustrate our main results, we treat the following examples.

Example 4.1. Let us consider the following problem:

$$\begin{cases} D^{\frac{\sqrt{3}}{2}}\left(D^{\frac{3}{4}}+\frac{1}{5e^4}\right)x(t)=f(t,x(t))+\sum\limits_{j=1}^{3}J^{\sigma_j}g_j(t,x(t)),t\in[0,1], \\ x(0)=\sum\limits_{l=1}^{q}c_l x(t_l),\ I^{\frac{1}{2}}x(1)=\sqrt{2}I^{\frac{1}{2}}x\left(\frac{1}{5}\right)+\frac{\sqrt{2}}{4}I^{\frac{1}{2}}x\left(\frac{5}{7}\right), \end{cases} \tag{4.1}$$

where $0<t_1<t_2<...<t_l<1, c_l, l=1,2,...,q,$ are given positive constants with $\sum\limits_{l=1}^{q}c_l<\frac{1}{2}$.

For this example, we have $\alpha=\frac{3}{4},\beta=\frac{\sqrt{3}}{2},\lambda=\frac{1}{5e^4},\delta_1=\frac{1}{2},\delta_2=\frac{1}{4},\delta_3=\frac{1}{8},p=\frac{1}{2},\mu_1=\sqrt{2},\mu_2=\frac{\sqrt{2}}{4},\eta_1=\frac{1}{5},\eta_2=\frac{5}{7}$ and $f(t,x)=\frac{x(t)}{35e^3\left(e^{1+t^2}+2\right)}$, $h_1(t,x)=\frac{\sin|x(t)|}{81}+\frac{1}{2}t, g_2(t,x)=\frac{\cos x(t)}{21(3^{2t}+5)}+\ln(t^2+1)$, and

$$g_3(t,x)=\frac{|x(t)|}{17(t+2)^3(|x(t)|+1)}+\arctan(2t+1).$$

Then we can find $\Lambda_1=6.03$, $\Lambda_2=1.4637\times10^{-2}$, $\Lambda_3=1.1652$.

Let $t\in[0,1]$, and $x,y\in\mathbb{R}$. Then

$$\begin{aligned} |f(t,x)-f(t,y)| &\leq \tfrac{1}{35e^3(e+2)}|x-y|, \\ |g_1(t,x)-g_1(t,y)| &\leq \tfrac{1}{81}|x-y|, \\ |g_2(t,x)-g_2(t,y)| &\leq \tfrac{1}{126}|x-y|, \\ |g_3(t,x)-g_3(t,y)| &\leq \tfrac{1}{136}|x-y|. \end{aligned}$$

Hence, the condition (A_1) holds with $N=\max\{N_j,j=1,...,4\}=\frac{1}{81}$, where $N_1=3.0148\times10^{-4}$, $N_2=\frac{1}{81}$, $N_3=\frac{1}{126}$, $N_4=\frac{1}{136}$. Also,we have

$$|h(x)-h(y)|=\left|\sum_{l=1}^{q}c_l x(t_l)-\sum_{l=1}^{q}c_l y(t_l)\right|\leq\sum_{l=1}^{q}c_l|x-y|.$$

So, (A_2) is satisfied with $M=\sum\limits_{l=1}^{q}c_l<\frac{1}{2}$.

We can show that

$$\Lambda_1 N+\Lambda_3 M=0.65704<1-\Lambda_2=0.98536.$$

Then, by Theorem 3.1, problem (4.1) has a unique solution.

Example 4.2. Consider the following problem:

$$\begin{cases} D^{\frac{1}{2}}\left(D^{\frac{2}{5}}+\frac{1}{17}\right)x(t)=f(t,x(t))+\sum_{j=1}^{2}J^{\sigma_j}g_j(t,x(t)),t\in[0,1], \\ x(0)=\frac{1}{15}+\frac{\ln 3}{19\pi}x(\xi),\ I^{\frac{1}{3}}x(1)=\frac{1}{14}I^{\frac{1}{3}}x\left(\frac{2}{3}\right)+\frac{2}{17}I^{\frac{1}{3}}x\left(\frac{1}{4}\right)+\frac{1}{19}I^{\frac{1}{3}}x\left(\frac{2}{5}\right). \end{cases} \tag{4.2}$$

Here, $\delta_1=\frac{3}{5},\delta_2=\frac{4}{9},x_0=\frac{2}{9},0<\xi<1$ and $f(t,x)=\frac{t\sin x(t)}{19}+\frac{2(t+1)|x|}{5|x|+1}$, $g_1(t,x(t))=e^t\sinh(t)\frac{\left(2-e^{3t^2}\right)x^2(t)-\frac{\sin(t)}{5}}{(1+e^t)(15|x|+t+11)}$, $g_2(t,x)=\frac{2\sin\left(\frac{x(t)}{2}\right)}{15\sqrt{\pi}+\cos^2 x(t)}+\frac{2+\cosh\left(\sqrt{\pi}t+1\right)}{3\sqrt{\pi}+3}$. With the given values, it is found that $\gamma(t)=\frac{(t+1)}{5},\gamma_1(t)=\frac{e^t\sinh(t)}{5(1+e^t)},\gamma_2(t)=\left(\frac{2+\cosh\left(\sqrt{\pi}t+1\right)}{15\sqrt{\pi}}\right),\psi(x)=\frac{|x|}{19}+2,\psi_1(x)=\left(\frac{|x|}{3}+1\right),\psi_2(x)=|x|+5,\ \Lambda_2=4.3588\times10^{-2},\Lambda_3=8.5861\times10^{-2},\Pi=2.1728,\Pi_1=1.4702,\ \Pi_2=1.6545$ and the condition

$$\frac{\delta_0}{\Pi\|\gamma\|\psi(\delta_0)+\sum_{j=1}^{2}\Pi_j\|\gamma_j\|\psi_j(\delta_0)+\Lambda_3|x_0|}>\frac{1}{1-(\Lambda_2+\Lambda_3M)},$$

implies that $\delta_0>32.355$. Clearly all the conditions of Theorem 3.3 are satisfied and hence by the conclusion of Theorem 3.3, the problem (4.2) has a solution on $[0,1]$.

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Department of Mathematics, Faculty of Science and Technology, UDBKM University, Algeria
Email address: houas_mohamed@yahoo.fr