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ITERATIVE SOLUTIONS FOR COMMON FIXED POINTS OF NONEXPANSIVE MAPPINGS AND STRONGLY PSEUDOCONTRACTIVE MAPPINGS WITH APPLICATIONS

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Abstract. In this paper, we introduce a mixed-type three-step iterative scheme for approximating common fixed points of nonexpansive mappings and strongly pseudocontractive mappings. Using our new iterative scheme, we state and prove a strong convergence result for approximating the common fixed points of nonexpasive mappings and strongly pseudocontractive mappings in a real Banach space. In addition, we give some examples of the mappings and prototypes of control parameters which satisfy the mild conditions used in our result. Numerical examples and applications are considered. Our results generalize and unify several well known results in the existing literature.

1. Introduction

Through out this paper, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}$ denotes the set real numbers. Let X be a real Banach space and $J : X \to 2^{X^*}$ denote the normalized duality mapping defined by

$$J(w) = \{f^* \in X^* : \langle w, f^* \rangle = \|w\|^2 = \|f^*\|^2\},\ \forall w \in X, \tag{1.1}$$

where X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing of X and X^*. In the sequel, we shall use j to denote the single-valued duality mapping and $F(H)$ to denote the set of fixed points of mapping H, i.e., $F(H) = \{w \in X : Hw = w\}$.

The following definition will be useful in the sequel.

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Definition 1.1. Let K be a nonempty closed convex subset of real Banach space X. A mapping $H : K \to K$ is said to be:

- *nonexpansive* if,
$$\|Hw - Hp\| \leq \|w - p\|, \ \forall w, p \in K; \tag{1.2}$$
- *strongly accretive* if there exist $j(w-p) \in J(w-p)$ and a constant $k \in (0,1)$ such that
$$\langle Hw - Hp, j(w-p)\rangle \geq k\|w-p\|^2, \ \forall w, p \in K; \tag{1.3}$$
- *strongly pseudocontractive* (Kim et al. [21]) if for all $w, p \in K$, there exists a constant $k \in (0,1)$ and $j(w-p) \in J(w-p)$ satisfying
$$\langle Hw - Hp, j(w-p)\rangle \leq k\|w-p\|^2. \tag{1.4}$$

The class of strongly pseudocontractive operators is known to be closely related to the class of strongly accretive operators (see [25]). Clearly, H is a strongly pseudocontractive operator if and only if $(I - H)$ is strongly accretive, where I denotes the identity mapping. These classes of mappings have been studied by several researchers in the area of nonlinear analysis (see, for example, [7, 8, 17, 18, 22, 23, 25, 29] and the references therein). An iterative method is a mathematical procedure that generates a sequence to approximate solutions for the problems. Finding the fixed point of a given operator is in this class of problems. Obtaining the analytic solutions of most problems in nonlinear analysis is a very tedious task. In most cases, it is impossible to obtain analytic solutions to such problems, thus, the need for iterative methods arises.

On the other hand, the fixed point theory is one of the fastest growing research area of nonlinear functional analysis. It has many applications in finding solutions of ordinary differential equations, partial differential equations, variational inequalities and zero of monotone operators. There are numerous applications of fixed points theory to applied sciences and engineering, for instance, complex graphical shapes such as fractals, were discovered as fixed points as contain in [4]. Recently, Abbas et al. [1] presented an application of fixed point iterative precess in generation of fractals namely Julia and Mandelbrot sets for the complex polynomials of the form $H(w) = w^n + mw + r$, where $m, r \in \mathbb{C}$ and $n \geq 2$. Fractals represent the phenomena of expanding or unfolding symmetries which exhibit similar patterns displayed at every scale. They proved some escape time results for generation of Julia and Mandelbrot sets via an iterative process. A visualization of the Julia and Mandelbrot set for certain complex polynomials and their graphical behaviors were examined. Further, they illustrated the effects of parameters on the color variation and shape of fractal.

Several problems emanating from applied sciences and engineering can expressed in the form of fixed point equations. Operator equations representing diverse area of study such as steady state temperature, distribution, chemical reactions, neutron transport theory, economy theories and epidemics, often require appropriate and adequate solutions. In order to find solutions of these equations, then one has to locate the fixed point and approximate its value. Iterative methods play vital roles in approximating such fixed points. It is well known that the celebrated Banach contraction principle no longer hold if the class of mappings is extended to the class of nonexpansive mappings which is a more general class of mappings. This implies that a nonexpansive mapping need not admit a fixed point on a complete metric space. Furthermore, the Picard iterative method does not really converge to the fixed point of a nonexpansive mapping in a complete metric space even when the existence of the fixed point is guaranteed. As a result of the limitation of Picard iterative method, several researchers in nonlinear analysis have been active in constructing new iterative schemes for the class of nonexpansive mappings and other more general class of mappings. Some well known iterative schemes are Mann [23], Ishikawa [16], Noor [26], Argawal et al. [3], normal S-iteration [32], Abass and Nazir [2], this just to mention but a few of the numerous iterative schemes in the existing literature.

In 2011, Sahu [32]-[33] introduced an iteration process known as the normal S-iteration process in Banach spaces and proved that it converges at a rate faster than the iteration processes in the existing literature. The normal S-iteration method have been kept a watchful eye since it was introduced in 2011. Many authors have studied and modified it in different versions. Most of these authors have equally shown that the normal S-iteration method converges faster both numerically and analytically than all of Picard [30], Krasnosel'skii [20], Mann [23], and Ishikawa [16] iteration processes for contraction mappings in the sense of Berinde [5]. The normal S-iteration process is defined as follows:

$$\begin{cases} w_0 \in K, \\ w_{n+1} = Hp_n, & \forall n \geq 1, \\ p_n = (1-\eta_n)w_n + \eta_n H w_n, \end{cases} \tag{1.5}$$

where $\{\eta_n\}$ is a sequence in [0,1].

The simplicity and fast rate of convergence of normal S-iteration process has enticed several authors to studying it as well as its modified forms, (see for example, [5, 17, 18, 32, 33] and the references in them). The normal S-iteration process has wide range of applications, for instance, it has been shown that it converge strongly to the solution of some nonlinear integral equations like the mixed type Volterra-Fredholm functional integral equations and delay differential equations, (see for examples, [14, 27, 28] and the references therein).

In [11], Debata and Dass introduced and studied the concept of two mappings in an iterative scheme. Since then, many researchers have found it interesting and there are so many papers in the existing literature in this direction for the past 24 years (see, for example, [12, 17, 18]).

Motivated by the performance of the normal S-iteration precess, Kang et al. [17] considered the following hybrid normal S-iteration process for approximating the common fixed point of nonexpansive mappings and strongly pseudocontractive mappings in a real Banach space.

$$\begin{cases} w_0 \in K, \\ w_{n+1} = Sp_n, & \forall n \geq 1, \\ p_n = (1-\eta_n)w_n + \eta_n H w_n, \end{cases} \tag{1.6}$$

where $\{\eta_n\}$ is a sequence in [0,1], S a nonexpansive mapping and H a strongly pseudocontractive mapping. In [17], Kang et al. used the following condition to prove their strong convergence result:

Remark 1.2. [17] Let $S, H : K \to K$ be two mappings. The mappings S and H are said to satisfy condition (C) if

$$\|w - Sp\| \leq \|Sw - Sp\|, \quad \|w - Hp\| \leq \|Hw - Hp\|, \tag{1.7}$$

for all $w, p \in K$.

It is well known that three-step iteration processes in most cases give better approximation than that of one-step (Mann iteration scheme) and two-step (Ishikawa iterative scheme) and many researchers have found three-step iterative processes more interesting in the past two decades since Noor [26] introduced the three-step iterative scheme. There are many recent papers on three steps iteration processes (see for example, [2, 6, 12, 13, 19, 24, 31] and the references there in).

Recently, Mebawondu and Mewomo [24] introduced the folowing three steps iteration process for approximating the fixed point of a contractive like mapping and Suzuki generalized nonexapansive mapping in the frame work of uniformly convex Banach space

$$\begin{cases} w_0 \in K, \\ w_{n+1} = Hp_n, \\ p_n = (1 - \eta_n - \mu_n)q_n + \eta_n Hq_n + \mu_n Hw_n, \\ q_n = (1 - r_n)w_n + r_n Hw_n, \end{cases} \quad \forall n \geq 1, \tag{1.8}$$

where η_n, μ_n and r_n are real sequences in [0,1].

Motivated and inspired by the above results, we introduce the following mixed-type three steps iterative scheme for approximating the common fixed points of nonexpansive mapping and strongly pseudocontractive mapping in a real Banach space

$$\begin{cases} w_0 \in K, \\ w_{n+1} = Sp_n, \\ p_n = (1 - \eta_n - \mu_n)q_n + \eta_n Hq_n + \mu_n Hw_n, \\ q_n = (1 - r_n)w_n + r_n Sw_n, \end{cases} \quad \forall n \geq 1, \tag{1.9}$$

where η_n, μ_n, r_n are real sequences in [0,1], S is a nonexpansive mapping and H a strongly pseudocontractive mapping.

Obviously, (1.9) reduces to:

- (1.5) when $\mu_n = r_n = 0$, $S = H$ or $\eta_n = r_n = 0$, $S = H$ or $\eta_n = \mu_n = 0$.
- (1.6) when $\mu_n = r_n = 0$ or $\eta_n = r_n = 0$.
- (1.8) when $S = H$.

Our new mixed type iteration process is independent of (1.5), (1.6) and (1.8), but properly includes them and several others in the existing literature.

The purpose of this paper is to prove the strong convergence of our new iterative scheme (1.9) to the common fixed points of nonexpansive mapping and strongly pseudocontractive mapping without the necessity of condition (C) as considered by Kang et al [17]. A numerical example will be given to demonstrate the convergence of our new iterative scheme (1.9) to the common fixed point of nonexpansive mapping and strongly pseudocontractive mapping. In addition, we will prove strong convergence of a special case of our new scheme (1.9) to the unique solution of a delay differential equation. This shows the applicability of our results. Hence, our results mainly improve, generalize and unify the results of Kang et al. [17], Sahu [32] and several others in the existing literature.

2. Preliminaries

In order to prove our main results, we also need the following lemmas.

Lemma 2.1 (see [17]). *Let $J : X \to 2^{X^*}$ be the normalized duality mapping. Then for any $w, p \in X$, one has*

$$\|w+p\|^2 \leq \|w\|^2 + 2\langle p, j(w+p)\rangle, \ \ \forall j(w+p) \in J(w+p). \tag{2.1}$$

Lemma 2.2. *Let $\{\theta_n\}$ and $\{\lambda_n\}$ be nonnegative real sequences satisfying the following inequalities:*

$$\theta_{n+1} \leq (1-\sigma_n)\theta_n + \lambda_n, \tag{2.2}$$

where $\sigma_n \in (0,1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \sigma_n = \infty$ and $\lim_{n\to\infty} \frac{\lambda_n}{\sigma_n} = 0$. Then $\lim_{n\to\infty} \theta_n = 0$.

Lemma 2.3. *Let $\{\theta_n\}$ and λ_n be nonnegative real sequences satisfying the following inequalities:*

$$\theta_{n+1} \leq (1-\sigma_n)\theta_n + \sigma_n\lambda_n, \tag{2.3}$$

where $\sigma_n \in (0,1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \sigma_n = \infty$ and $\lambda_n \geq 0$ for all $n \in \mathbb{N}$. Then

$$0 \leq \limsup_{n\to\infty} \theta_n \leq \limsup_{n\to\infty} \lambda_n. \tag{2.4}$$

3. Main results

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Banach space X. Let $S : K \to K$ be a nonexpansive mapping and let $H : K \to K$ be a strongly pseudocontractive mapping with bounded range bounded. Furthermore, let H be uniformly continuous. Suppose that $\Im = F(S) \bigcap F(H) = \{w \in K : Sw = Hw = w\} \neq \emptyset$ and let $\{\eta_n\}$, $\{\mu_n\}$, $\{r_n\}$ be real sequences in [0,1] such that $\eta_n + \mu_n \leq 1$. If the following conditions are satisfied:*

- (i) $\lim_{n\to\infty} (\eta_n + \mu_n) = 0 = \lim_{n\to\infty} r_n$;
- (ii) $\sum_{n=1}^{\infty} (\eta_n + \mu_n) = \infty$.

For arbitrary $w_0 \in K$, let $\{w_n\}$ be the sequence iteratively defined by (1.9). Then the sequence $\{w_n\}$ converges strongly to a point in $\Im$.

Proof. Fix $z \in \Im$. Since H has a bounded range, we let

$$M_1 = \|w_0 - z\| + \sup_{n\geq 1} \|Hq_n - z\| + \sup_{n\geq 1} \|Hw_n - z\|. \tag{3.1}$$

It is easy to see that $M_1 < \infty$. Inductively, it follows from (3.1) that $\|w_0 - z\| \leq M_1$. Next, we let $\|w_n - z\| \leq M_1$. We nextnow show that $\|w_{n+1} - z\| \leq M_1$. Since S is a nonexpansive mapping, we obtain

$$\begin{aligned}
\|q_n - z\| &= \|(1 - r_n)(w_n - z) + r_n(Sw_n - z)\| \\
&\leq (1 - r_n)\|w_n - z\| + r_n\|Sw_n - z\| \\
&\leq (1 - r_n)\|w_n - z\| + r_n\|w_n - z\| \\
&= \|w_n - z\|.
\end{aligned} \tag{3.2}$$

Using (1.9), (3.1) and (3.2), we have that

$$\begin{aligned}
\|w_{n+1} - z\| &\leq \|Sp_n - Sz\| \\
&\leq \|p_n - z\| \\
&= \|(1 - \eta_n - \mu_n)q_n + \eta_n Hq_n + \mu_n Hw_n - z\| \\
&= \|(1 - \eta_n - \mu_n)(q_n - z) + \eta_n(Hq_n - z) + \mu_n(Hw_n - z)\| \\
&\leq (1 - \eta_n - \mu_n)\|q_n - z\| + \eta_n\|Hq_n - z\| + \mu_n\|Hw_n - z\| \\
&\leq (1 - \eta_n - \mu_n)\|w_n - z\| + \eta_n\|Hq_n - z\| + \mu_n\|Hw_n - z\| \\
&\leq (1 - \eta_n - \mu_n)M_1 + \eta_n\|Hq_n - z\| + \mu_n\|Hw_n - z\| \\
&= (1 - \eta_n - \mu_n)[\|w_0 - z\| + \sup_{n\geq 1}\|Hq_n - z\| + \sup_{n\geq 1}\|Hw_n - z\|] \\
&\quad +\eta_n\|Hq_n - z\| + \mu_n\|Hw_n - z\| \\
&\leq \|w_0 - z\| + (1 - \eta_n - \mu_n)\sup_{n\geq 1}\|Hq_n - z\| \\
&\quad +(1 - \eta_n - \mu_n)\sup_{n\geq 1}\|Hw_n - z\| + \eta_n\|Hq_n - z\| + \mu_n\|Hw_n - z\| \\
&\leq \|w_0 - z\| + [(1 - \eta_n)\sup_{n\geq 1}\|Hq_n - z\| + \eta_n\|Hq_n - z\|] \\
&\quad +[(1 - \mu_n)\sup_{n\geq 1}\|Hw_n - z\| + \mu_n\|Hw_n - z\|] \\
&\leq \|w_0 - z\| + [(1 - \eta_n)\sup_{n\geq 1}\|Hq_n - z\| + \eta_n\sup_{n\geq 1}\|Hq_n - z\|] \\
&\quad +[(1 - \mu_n)\sup_{n\geq 1}\|Hw_n - z\| + \mu_n\sup_{n\geq 1}\|Hw_n - z\|] \\
&= \|w_0 - z\| + \sup_{n\geq 1}\|Hq_n - z\| + \sup_{n\geq 1}\|Hw_n - z\| \\
&= M_1.
\end{aligned}$$

From the demonstration above, we know that the sequence $\{\|w_n - z\|\}_{n\geq 0}$ is bounded. Since S is uniformly continuous, then we have that $\{\|Sw_n - z\|\}_{n\geq 1}$ is also bounded. It follows from (3.2) that $\{\|q_n - z\|\}_{n\geq 0}$ is bounded. Thus, there exists a constant M_2 such that

$$M_2 = \sup_{n\geq 1}\|w_n - z\| + \sup_{n\geq 1}\|q_n - z\| + \sup_{n\geq 1}\|Sw_n - z\| + M_1. \tag{3.3}$$

Using (1.9), we obtain that

$$\begin{aligned}
\|p_n - w_n\| &\leq \|p_n - q_n\| + \|q_n - w_n\| \\
&= \|(1-\eta_n-\mu_n)q_n + \eta_n Hq_n + \mu_n Hw_n - q_n\| \qquad (3.4) \\
&\quad +\|(1-r_n)w_n + r_n Sw_n - w_n\| \\
&= \|\eta_n(Hq_n - q_n) + \mu_n(Hw_n - q_n)\| + \|r_n(Sw_n - w_n)\| \\
&\leq \eta_n(\|Hq_n - z\| + \|z - q_n\|) + \mu_n(\|Hw_n - z\| + \|z - q_n\|) \\
&\quad +r_n(\|Sw_n - z\| + \|z - w_n\|) \\
&\leq 2M(\eta_n + \mu_n + r_n) \qquad (3.5)
\end{aligned}$$

Thus, from condition (i) and (3.5) we obtain

$$\lim_{n\to\infty} \|p_n - w_n\| = 0. \qquad (3.6)$$

This implies that $\{\|p_n - w_n\|\}$ is bounded. Let

$$M_3 = \sup_{n\geq 1} \|p_n - w_n\| + M_2.$$

Since

$$\|p_n - z\| \leq \|p_n - w_n\| + \|w_n - z\| \leq M_3,$$

we have that $\{\|p_n - z\|\}_{n\geq 0}$ is bounded. Set

$$M_4 = \sup_{n\geq 1} \|p_n - z\| + \sup_{n\geq 1} \|Hp_n - z\| + M_3. \qquad (3.7)$$

Denote

$$M = M_1 + M_2 + M_3 + M_4, \qquad (3.8)$$

obviously, $M < \infty$. Now, using (1.9) for all $n \geq 1$, we obtain

$$\|w_{n+1} - z\|^2 = \|Sp_n - z\|^2 = \|Sp_n - Sz\|^2 \leq \|p_n - z\|^2, \qquad (3.9)$$

thus, from (1.9), (3.2) and Lemma 2.1, we have

$$\begin{aligned}
\|p_n - z\|^2 &= \|(1-\eta_n-\mu_n)(q_n - z) + \eta_n(Hq_n - z) \\
&\quad +\mu_n(Hw_n - z)\|^2 \\
&\le (1-\eta_n-\mu_n)^2\|q_n - z\|^2 + 2\langle \eta_n(Hq_n - z) \\
&\quad +\mu_n(H\eta_n - z), j(p_n - z)\rangle \\
&\le (1-\eta_n-\mu_n)^2\|w_n - z\|^2 + 2\langle \eta_n(Hq_n - z) \\
&\quad +\mu_n H(w_n - z), j(p_n - z)\rangle \\
&\le (1-\eta_n-\mu_n)^2\|w_n - z\|^2 + 2\eta_n\langle Hq_n - z, j(p_n - z)\rangle \\
&\quad +2\mu_n\langle Hw_n - z, j(p_n - z)\rangle \\
&= (1-\eta_n-\mu_n)^2\|w_n - z\|^2 \\
&\quad +2\eta_n\langle Hq_n - Hp_n, j(p_n - z)\rangle + 2\eta_n\langle Hp_n - z, j(p_n - z)\rangle \\
&\quad +2\mu_n\langle Hw_n - Hp_n, j(p_n - z)\rangle + 2\mu_n\langle Hp_n - z, j(p_n - z)\rangle \\
&\le (1-\eta_n-\mu_n)^2\|w_n - z\|^2 + 2\eta_n\|Hq_n - Hp_n\|\|p_n - z\| \\
&\quad +2\mu_n\|Hw_n - Hp_n\|\|p_n - z\| + 2(\eta_n+\mu_n)\langle Hp_n - z, j(p_n - z)\rangle \\
&\le (1-\eta_n-\mu_n)^2\|w_n - z\|^2 + 2\eta_n\xi_n + 2\mu_n\gamma_n \\
&\quad +2(\eta_n+\mu_n)\langle Hp_n - z, j(p_n - z)\rangle \\
&\le (1-\eta_n-\mu_n)^2\|w_n - z\|^2 + 2(\eta_n+\mu_n)\max(\xi_n,\gamma_n) \\
&\quad +2(\eta_n+\mu_n)\langle Hp_n - z, j(p_n - z)\rangle, \qquad (3.10)
\end{aligned}$$

where

$$\xi_n = M\|Hq_n - Hp_n\|, \gamma_n = M\|Hw_n - Hp_n\|.$$

Since H is uniformly continuous, it follows from (3.6) that

$$\lim_{n\to\infty} \|Hw_n - Hp_n\| = 0. \qquad (3.11)$$

Thus, we obtain

$$\lim_{n\to\infty} \gamma_n = 0. \qquad (3.12)$$

Again, from (1.9), we obtain

$$\begin{aligned}
\|q_n - w_n\| &= \|(1-r_n)w_n + r_nSw_n - w_n\| \\
&= r_n\|Sw_n - w_n\| \\
&\le r_n(\|Sw_n - z\| + \|z - w_n\|) \\
&\le 2r_nM_2. \qquad (3.13)
\end{aligned}$$

It follows from (3.13) and condition (i) that

$$\lim_{n\to\infty} \|q_n - w_n\| = 0. \qquad (3.14)$$

Note that

$$\begin{aligned}
\|q_n - p_n\| &= \|q_n - w_n + w_n - p_n\| \\
&\le \|q_n - w_n\| + \|w_n - p_n\|. \qquad (3.15)
\end{aligned}$$

From (3.6) and (3.14), it follows from (3.15) that

$$\lim_{n\to\infty} \|q_n - p_n\| = 0. \tag{3.16}$$

Since H is uniformly continuous, we have

$$\lim_{n\to\infty} \|Hq_n - Hp_n\| = 0. \tag{3.17}$$

It follows that

$$\lim_{n\to\infty} \xi_n = 0. \tag{3.18}$$

Since H is a strongly pseudocontractive map, it follows from (3.10) that

$$\begin{aligned} \|p_n - z\|^2 &\leq (1-\eta_n-\mu_n)^2\|w_n - z\|^2 \\ &\quad +2(\eta_n+\mu_n)\max(\xi_n,\gamma_n) \\ &\quad +2(\eta_n+\mu_n)k\|p_n - z\|^2 \\ &\leq \frac{(1-\eta_n-\mu_n)^2}{1-2(\eta_n+\mu_n)k}\|w_n - z\|^2 \qquad (3.19) \\ &\quad +\frac{2(\eta_n+\mu_n)\max(\xi_n,\gamma_n)}{1-2(\eta_n+\mu_n)k}. \qquad (3.20) \end{aligned}$$

Since $\eta_n + \mu_n \to 0$ as $n \to \infty$, there exists a positive integer $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\eta_n + \mu_n \leq \min\left\{\frac{1}{4k}, \frac{1-k}{(1-k)^2+k^2}\right\}, \tag{3.21}$$

where $k < \frac{1}{2}$. This implies that $\frac{1-(\eta_n+\mu_n)}{1-2(\eta_n+\mu_n)k} \leq 1$ and $\frac{1}{1-2(\eta_n+\mu_n)k} \leq \frac{1}{2}$. It now follows from (3.20) that

$$\|p_n - z\|^2 \leq [1-(\eta_n+\mu_n)]\|w_n - z\|^2 + 4(\eta_n+\mu_n)\max(\xi_n,\gamma_n). \tag{3.22}$$

Now, with the help of (3.12) and (3.18), we have $\max(\xi_n, \gamma_n) \to 0$ as $n \to \infty$. It follows from (3.9) and (3.22) that

$$\|w_{n+1} - z\|^2 \leq [1-(\eta_n+\mu_n)]\|w_n - z\|^2 + 4(\eta_n+\mu_n)\max(\xi_n,\gamma_n). \tag{3.23}$$

For all $n \geq 1$, set

$$\begin{aligned} \theta_n &= \|w_n - z\| \\ \sigma_n &= \eta_n + \mu_n \\ \lambda_n &= 4\max(\xi_n,\gamma_n). \end{aligned}$$

Therefore, all the conditions of Lemma 2.1 are satisfied. Hence,

$$\lim_{n\to\infty} \|w_n - z\| = 0.$$

This completes the proof of Theorem 3.1. □

The following results are obtain immediately from Theorem 3.1.

Corollary 3.2. *Let K be a nonempty closed convex subset of a real Banach space X. Let $S : K \to K$ be a nonexpansive mapping and let $H : K \to K$ be a strongly pseudocontractive mapping with bounded range. Furthermore, let H be uniformly continuous. Suppose that $\Im = F(S) \bigcap F(H) = \{w \in K : Sw = Hw = w\} \neq \emptyset$ and let $\{\eta_n\}$, $\{r_n\}$ be real sequences in [0,1] satisfying the following conditions:*

(i) $\lim\limits_{n\to\infty} \eta_n = \lim\limits_{n\to\infty} r_n = 0$;

(ii) $\sum\limits_{n=1}^{\infty} \eta_n = \infty$.

For arbitrary $w_0 \in K$, let $\{w_n\}$ be the sequence iteratively defined by

$$\begin{cases} w_0 \in K, \\ w_{n+1} = Sp_n, \\ p_n = (1-\eta_n)q_n + \eta_n Hq_n, \\ q_n = (1-r_n)w_n + r_n Sw_n. \end{cases} \quad \forall n \geq 1, \tag{3.24}$$

Then the sequence $\{w_n\}$ converges strongly to a point in $\Im$.

Proof. Putting $\mu_n = 0$ in Theorem 3.1, we have the desired conclusion. □

Corollary 3.3. *Let K be a nonempty closed convex subset of a real Banach space X. Let $S : K \to K$ be a nonexpansive mapping and let $H : K \to K$ be a strongly pseudocontractive mapping with bounded range. Furthermore, let H be uniformly continuous. Suppose that $\Im = F(S) \bigcap F(H) = \{w \in K : Sw = Hw = w\} \neq \emptyset$ and let $\{\mu_n\}$, $\{r_n\}$ be real sequences in [0,1] satisfying the following conditions:*

(i) $\lim\limits_{n\to\infty} \mu_n = \lim\limits_{n\to\infty} r_n = 0$;

(ii) $\sum\limits_{n=1}^{\infty} \mu_n = \infty$.

For arbitrary $w_0 \in K$, let $\{w_n\}$ be the sequence iteratively defined by

$$\begin{cases} w_0 \in K, \\ w_{n+1} = Sp_n, \\ p_n = (1-\mu_n)q_n + \mu_n Hw_n, \\ q_n = (1-r_n)w_n + r_n Sw_n. \end{cases} \quad \forall n \geq 1, \tag{3.25}$$

Then the sequence $\{w_n\}$ converges strongly to a point in $\Im$.

Proof. Putting $\eta_n = 0$ in Theorem 3.1, we have the following result. □

Corollary 3.4. *Let K be a nonempty closed convex subset of a real Banach space X. Let $S : K \to K$ be a nonexpansive mapping and let $H : K \to K$ be a strongly pseudocontractive mapping with bounded range. Furthermore, let H be uniformly continuous. Suppose that $\Im = F(S) \bigcap F(H) = \{\eta \in K : Sw = Hw = w\} \neq \emptyset$ and let $\{\eta_n\}$ and $\{\mu_n\}$ be real sequences in [0,1] such that $\eta_n + \mu_n \leq 1$. If the following conditions are satisfied:*

(i) $\lim\limits_{n\to\infty} (\eta_n + \mu_n) = 0$;

(ii) $\sum\limits_{n=1}^{\infty} (\eta_n + \mu_n) = \infty$.

For arbitrary $\eta_0 \in C$, *let* $\{\eta_n\}$ *be the sequence iteratively defined by*

$$\begin{cases} w_0 \in K, \\ w_{n+1} = Sp_n, & \forall n \geq 1. \\ p_n = (1-\eta_n-\mu_n)w_n + \eta_n Hw_n + \mu_n Hw_n. \end{cases} \tag{3.26}$$

Then the sequence $\{w_n\}$ *converges strongly to a point in* $\Im$.

Proof. Setting $r_n = 0$ in Theorem 3.1, we have the following result. □

Corollary 3.5. *Let* K *be a nonempty closed convex subset of a real Banach space* X. *Let* $S : K \to K$ *be a nonexpansive mapping and let* $H : K \to K$ *be a strongly pseudocontractive mapping with with bounded range. Furthermore, let* H *be uniformly continuous. Suppose that* $\Im = F(S) \bigcap F(H) = \{w \in K : Sw = Hw = w\} \neq \emptyset$ *and let* $\{\mu_n\}$ *be a real sequence in [0,1] satisfying the following conditions:*

(i) $\lim_{n\to\infty} \mu_n = 0$;

(ii) $\sum_{n=1}^{\infty} \mu_n = \infty$.

For arbitrary $w_0 \in K$, *let* $\{w_n\}$ *be the sequence iteratively defined by*

$$\begin{cases} w_0 \in K, \\ w_{n+1} = Sp_n, & \forall n \geq 1, \\ p_n = (1-\mu_n)w_n + \mu_n Hw_n, \end{cases} \tag{3.27}$$

Then the sequence $\{w_n\}$ *converges strongly to a point in* $\Im$.

Proof. Setting $\eta_n = 0$ in Corollary 3.4, we have the following result. □

Corollary 3.6. *Let* K *be a nonempty closed convex subset of a real Banach space* X. *Let* $H : K \to K$ *be a strongly pseudocontractive mapping with bounded range. Furthermore, let* H *be uniformly continuous. Suppose that* $F(H) = \{\eta \in K : Hw = w\} \neq \emptyset$ *and let* $\{\eta_n\}$ *and* $\{\mu_n\}$ *be real sequences in [0,1] such that* $\eta_n + \mu_n \leq 1$. *If the following conditions are satisfied:*

(i) $\lim_{n\to\infty} (\eta_n + \mu_n) = 0$;

(ii) $\sum_{n=1}^{\infty} (\eta_n + \mu_n) = \infty$.

For arbitrary $\eta_0 \in C$, *let* $\{\eta_n\}$ *be the sequence iteratively defined by*

$$\begin{cases} w_0 \in K, & \\ w_{n+1} = (1-\eta_n-\mu_n)w_n + \eta_n Hw_n + \mu_n Hw_n. & \forall n \geq 1. \end{cases} \tag{3.28}$$

Then the sequence $\{w_n\}$ *converges strongly to a point in* $F(H)$.

Proof. Setting $S = I$ in Corollary 3.4, we have the following result. □

Corollary 3.7. *Let* K *be a nonempty closed convex subset of a real Banach space* X. *Let* $H : K \to K$ *be a strongly pseudocontractive mapping with bounded range. Furthermore, let* H *be uniformly continuous. Suppose that* $F(H) = \{w \in K : Hw = w\} \neq \emptyset$ *and let* $\{p_n\}$ *be a real sequence in [0,1] satisfying the following conditions:*

(i) $\lim_{n\to\infty} \mu_n = 0$;

(ii) $\sum_{n=1}^{\infty} \mu_n = \infty$.

For arbitrary $w_0 \in K$*, let* $\{w_n\}$ *be the sequence iteratively defined by*

$$\begin{cases} w_0 \in C, \\ w_{n+1} = (1-\mu_n)w_n + \mu_n H w_n, \end{cases} \quad \forall n \geq 1, \tag{3.29}$$

Then $\{w_n\}$ *converges strongly to a point in* $F(H)$*.*

Proof. Setting $S = I$ in Corollary 3.5, we have the following result. □

Remark 3.8.
(1) Obviously, the result of Kang et al. [17] is captured in Corollary 3.5.
(2) The results of Kang et al. [17] is special a case of Theorem 3.1. Hence our result generalize and improve the result of Kang et al [17] and several others in the existing literature.
(1) We prove our strong convergence theorem without using condition (C), which is used by Kang et al. [17] to prove their result.

4. The numerical example

In this section, we give some numerical examples of the mappings and Prototype of the control parameters considered in our theorem and then show that our new iteration process converges to the common fixed points of these mappings.

Example 4.1. Let X be the real line with the usual norm $|\cdot|$ and $K = [0,\infty)$. Define $S, H : K \to K$ by

$$Hw = \frac{w}{(1+w)}, \quad Sw = \frac{w}{2}, \forall w \in K.$$

Then the following are satisfied:

(i) H is a strongly pseudocontractive mapping which is uniformly continuous on K with range bounded in $[0,1)$ and S a nonexpansive mapping which is uniformly continuous on K.

(ii) Obviously, $H(0) = 0$, $S(0) = 0$, that is, 0 is the common fixed point of H and S, i.e., $\Im = F(H) \bigcap F(S) = \{0\}$.

Put $\eta_n = \mu_n = \frac{1}{n+2}$, $r_n = \frac{1}{n+1}$. Observe that all the conditions of Theorem 3.1 are satisfied. Then for arbitrary $w_0 \in K$, our new scheme (1.9) convergences to the common fixed point of H and S, which is $\{0\}$. Hence, Theorem 3.1 is applicable.

n	$w_0 = 0.5$	$w_0 = 1.0$
1	0.5000000000	1.0000000000
2	0.4133692366	0.7757936508
3	0.0438482921	0.0617984402
4	0.0054433573	0.0060616887
5	0.0006904460	0.0006096284
6	0.0000878196	0.0000614678
7	0.0000111739	0.0000061993
8	0.0000014218	0.0000006252
9	0.0000001809	0.0000000631
10	0.0000000230	0.0000000064
11	0.0000000029	0.0000000006
12	0.0000000004	0.0000000001
13	0.0000000000	0.0000000000

TABLE 1. The convergence of mixed type three-step iterative scheme defined by (1.9)

5. Applications to a Delay Differential Equation

Delay differential equations are used in many physical phenomena of interest in biology, medicine, chemistry, physics, engineering, economics, among others (see for example, [9], [10], [34], and the references there in). It is our purpose in this section to exhibit the applicability of a special case of our newly introduced mixed type three-step iteration process (1.9) which is defined for $H = S$ as follows:

$$\begin{cases} w_0 \in K, \\ w_{n+1} = Hp_n, \\ p_n = (1 - \eta_n - \mu_n)q_n + \eta_n Hq_n + \mu_n Hw_n, \\ q_n = (1 - r_n)w_n + r_n Hw_n. \end{cases} \quad \forall n \geq 1, \tag{5.1}$$

Let $C([d, e])$ stand for the space of all continuous real-valued functions on a closed interval $[d, e]$ endowed which is endowed with the Chebyshev norm $\|w - p\|_\infty = \max_{g\in[d,e]} |w(g) - p(g)|$. Then $(C([d, e]), \|\cdot\|_\infty)$ is generally known to be a Banach space, with maximum norm

$$\|w - p\|_\infty = \max_{g\in[d,e]} \|w(g) - p(g)\|_\infty, \forall w, \upsilon \in C([d, e]), \tag{5.2}$$

see [15]. Our interest now is to consider the following delay differential equation,

$$w'(g) = f(g, w(g), w(g - \tau)), \;\; g \in [g_0, e] \tag{5.3}$$

with initial condition

$$w(g) = \zeta(g), g \in [g_0 - \tau, g_0]. \tag{5.4}$$

We assume that the following conditions are satisfied

(T_1) $g_0, e \in \mathbb{R}, \tau > 0$;
(T_2) $f \in C([g_0, e] \times \mathbb{R}^2, \mathbb{R})$;
(T_3) $\zeta \in C([g_0 - \tau, e], \mathbb{R})$;

(T_4) there exists $L_f > 0$ such that

$$|f(g, u_1, u_2) - f(g, v_{1,2})| \leq L_f(|u_1 - v_1| + |u_2 - v_2|), \tag{5.5}$$

for all $u_1, u_2, v_1, v_2 \in R$ and $g \in [g_0, e]$;

(T_5) $2L_f(e - g_0) < 1$.

The problem (5.3)– (5.4) can be reformulated in the following integral equation:

$$w(g) = \begin{cases} \zeta(g), & g \in [g_0 - \tau, g_0], \\ \zeta(g_o) + \int_{g_0}^{g} f(s, w(s), w(s-\tau))ds, & g \in [g_0, b]. \end{cases} \tag{5.6}$$

Coman et al. [9] obtained the following results.

Theorem 5.1. *If conditions* $(T_1) - (T_5)$ *are satisfied. Then the problem* (5.3)–(5.4) *has a unique solution,* $z \in C([g_0 - \tau, e], \mathbb{R}) \cap C^1([g_0, e], \mathbb{R})$ *and*

$$z = \lim_{n\to\infty} H^n(w) \ \textit{for any} \ w \in ([g_0 - \tau, e], \mathbb{R}). \tag{5.7}$$

Now, we are ready to prove the strong convergence of (5.1) to the unique solution of delay differential equation.

Theorem 5.2. *Suppose that conditions* $(T_1) - (T_5)$ *are satisfied. Then the iterative sequence* $\{w_n\}$ *generated by iteration process* (5.1) *with real sequences* η_n, μ_n, r_n *in [0,1] such that* $\sum_{n=1}^{\infty}(\eta_n + \mu_n) = \infty$*, converges strongly to the unique solution of problem* (5.3)–(5.4)*, say* $z \in C([g_0 - \tau, b], \mathbb{R}) \cap C^1([g_0, e], \mathbb{R})$.

Proof. Let $\{w_n\}$ be an iterative sequence generated by the iteration process (5.1) for an operator defined by

$$Hw(e) = \begin{cases} \zeta(g), & d \in [g_0 - \tau, g_0], \\ \zeta(g_o) + \int_{g_0}^{g} f(s, w(s), w(s-\tau))ds, & g \in [g_0, e]. \end{cases} \tag{5.8}$$

Let z be a fixed point H. We will prove that $w_n \to z$ as $n \to \infty$. Apparently, it is easy to see that $w_n \to z$ as $n \to \infty$, for $g \in [g_0 - \tau, g_0]$. For $g \in [g_0, e]$, we have □

$$
\begin{aligned}
& \|q_n - z\|_\infty \\
= \; & \|(1-r_n)w_n + r_n H w_n - z\|_\infty \qquad (5.9)\\
= \; & \|(1-r_n)(w_n - z) + r_n H(w_n - z)\|_\infty \\
\leq \; & (1-r_n)\|w_n - z\|_\infty + r_n\|Hw_n - Hz\|_\infty \\
= \; & (1-r_n)\|w_n - z\|_\infty + r_n \max_{g\in[g_0-\tau,e]} |Hw_n(g) - Hz(g)| \\
= \; & (1-r_n)\|w_n - z\|_\infty + r_n \max_{g\in[g_0-\tau,e]} |\zeta(g_o) \\
& + \int_{g_0}^{g} f(s, w_n(s), w_n(s-\tau))ds - \zeta(g_o) - \int_{g_0}^{g} f(s, z(s), z(s-\tau))ds\Big| \\
= \; & (1-r_n)\|w_n - z\|_\infty + r_n \max_{g\in[g_0-\tau,e]} \left|\int_{g_0}^{g} f(s, w_n(s), w_n(s-\tau))ds \right.\\
& \left. - \int_{g_0}^{g} f(s, z(s), z(s-\tau))ds\right| \\
\leq \; & (1-r_n)\|w_n - z\|_\infty + r_n \max_{g\in[g_0-\tau,e]} \int_{g_0}^{g} |f(s, w_n(s), w_n(s-\tau)) \\
& - \int_{g_0}^{g} f(s, z(s), z(s-\tau))|ds \\
\leq \; & (1-r_n)\|w_n - z\|_\infty + r_n \max_{g\in[g_0-\tau,e]} \int_{g_0}^{g} L_f(|w_n(s) - z(s)| \\
& + |w_n(s-\tau) - z(s-\tau)|)ds \\
\leq \; & (1-r_n)\|w_n - z\|_\infty + r_n \int_{g_0}^{g} L_f(\max_{g\in[g_0-\tau,e]} |w_n(s) - z(s)| \\
& + \max_{g\in[g_0-\tau,e]} |w_n(s-\tau) - z(s-\tau)|)ds \\
\leq \; & (1-r_n)\|w_n - z\|_\infty + r_n \int_{g_0}^{g} L_f(\|w_n - z\|_\infty + \|w_n - z\|_\infty)ds \\
\leq \; & (1-r_n)\|w_n - z\|_\infty + 2r_n L_f(e - g_0)\|w_n - z\|_\infty \\
= \; & [1 - r_n(1 - 2L_f(e - g_0))]\|w_n - z\|_\infty. \qquad (5.10)
\end{aligned}
$$

Using (T_4), we have

$$
\begin{aligned}
& \|Hq_n - z\|_\infty \\
= \; & \|Hq_n - Hz\|_\infty = \max_{g\in[g_0-\tau,e]} |Hq_n(g) - Hz(g)| \qquad (5.11)\\
= \; & \max_{g\in[g_0-\tau,e]} |\zeta(g_0) \\
& + \int_{d_0}^{g} f(s, q_n(s), q_n(s-\tau))ds - \zeta(g_0) - \int_{g_0}^{g} f(s, z(s), z(s-\tau))ds\Big| \\
= \; & \max_{g\in[g_0-\tau,e]} \left|\int_{g_0}^{g} f(s, q_n(s), q_n(s-\tau))ds - \int_{g_0}^{g} f(s, z(s), z(s-\tau))ds\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \max_{g\in[g_0-\tau,e]} \int_{g_0}^{g} |f(s,q_n(s),q_n(s-\tau)) - \int_{g_0}^{g} f(s,z(s),z(s-\tau))|ds \qquad (5.12)\\
&\leq \max_{g\in[g_0-\tau,e]} \int_{g_0}^{g} L_f(|q_n(s)-z(s)| + |q_n(s-\tau)-z(s-\tau)|)ds\\
&\leq \int_{g_0}^{g} L_f(\max_{g\in[g_0-\tau,e]} |q_n(s)-z(s)| + \max_{g\in[g_0-\tau,e]} |q_n(s-\tau)-z(s-\tau)|)ds\\
&\leq \int_{g_0}^{g} L_f(\|q_n-z\|_\infty + \|q_n-z\|_\infty)ds\\
&\leq 2L_f(e-g_0)\|q_n-z\|_\infty. \qquad (5.13)
\end{aligned}
$$

Again, using (T_4), we obtain

$$
\begin{aligned}
& \|Hw_n - Hz\|_\infty\\
&= \max_{g\in[g_0-\tau,e]} |Hw_n(g) - Hz(g)| \qquad (5.14)\\
&= \max_{g\in[g_0-\tau,e]} |\zeta(g_o)\\
&\quad + \int_{g_0}^{g} f(s,w_n(s),w_n(s-\tau))ds - \zeta(g_o) - \int_{g_0}^{g} f(s,z(s),z(s-\tau))ds\Big|\\
&= \max_{z\in[z_0-\tau,e]} \left|\int_{g_0}^{g} f(s,w_n(s),w_n(s-\tau))ds - \int_{g_0}^{g} f(s,z(s),z(s-\tau))ds\right|\\
&\leq \max_{d\in[d_0-\tau,b]} \int_{g_0}^{d} |f(s,w_n(s),w_n(s-\tau)) - \int_{g_0}^{g} f(s,z(s),z(s-\tau))|ds\\
&\leq \max_{g\in[g_0-\tau,e]} \int_{g_0}^{g} L_f(|w_n(s)-z(s)| + |w_n(s-\tau)-z(s-\tau)|)ds\\
&\leq \int_{g_0}^{g} L_f(\max_{g\in[g_0-\tau,e]} |w_n(s)-z(s)| + \max_{g\in[g_0-\tau,e]} |w_n(s-\tau)-z(s-\tau)|)ds\\
&\leq \int_{g_0}^{g} L_f(\|w_n-z\|_\infty + \|q_n-z\|_\infty)ds\\
&\leq 2L_f(e-g_0)\|w_n-z\|_\infty. \qquad (5.15)
\end{aligned}
$$

Again, from (5.1), we have

$$
\begin{aligned}
\|p_n - z\| &\leq (1-\eta_n-\mu_n)\|q_n-z\|_\infty + \eta_n\|Hq_n - Hz\|_\infty\\
&\quad + \mu_n\|Hw_n - Hz\|_\infty. \qquad (5.16)
\end{aligned}
$$

Using (5.13) and (5.15) in (5.16), we obtain

$$
\begin{aligned}
\|p_n - z\| &\leq (1-\eta_n-\mu_n)\|q_n-z\|_\infty + 2\eta_n L_f(e-g_0)\|q_n-z\|\\
&\quad + 2\mu_n L_f(e-g_0)\|w_n-z\|_\infty\\
&= (2\eta_n L_f(e-g_0) + 1 - \eta_n - \mu_n)\|q_n-z\| + 2\mu_n L_f(e-g_0)\|w_n-z\|_\infty.
\end{aligned}
$$

Substituting (5.10) into (5.17), we obtain

$$\begin{aligned} \|p_n - z\| &\leq (2\eta_n L_f(e-g_0) + 1 - \eta_n - \mu_n)[1 - r_n(1 - 2L_f(e-g_0))]\|w_n - z\|_\infty \\ &\quad +2\mu_n L_f(e-g_0)\|w_n - z\|_\infty \\ &= \{(2\eta_n L_f(e-g_0) + 1 - \eta_n - \mu_n)[1 - r_n(1 - 2L_f(e-g_0))] \\ &\quad +2\mu_n L_f(e-g_0)\}\|w_n - z\|_\infty \\ &= [1 - \eta_n - \mu_n + 2\eta_n L_f(e-g_0) + 2\mu_n L_f(e-g_0) \\ &\quad -r_n(1 - 2L_f(e-g_0))(1 + 2\eta_n L_f(e-g_0) - \eta_n - \mu_n)]\|w_n - z\|_\infty. \end{aligned} \tag{5.17}$$

Since $\eta_n, \mu_n, r_n \in [0,1]$ and recalling from (T_5) that $2L_f(e-g_0) < 1$, we have that (5.17) reduces to

$$\begin{aligned} \|p_n - z\| &\leq [1 - \eta_n - \mu_n + 2\eta_n L_f(e-g_0) + 2\mu_n L_f(e-g_0)]\|w_n - z\|_\infty \\ &= [1 - (\eta_n + \mu_n)(1 - 2L_f(e-g_0))]\|w_n - z\|_\infty. \end{aligned} \tag{5.18}$$

From (5.1), we obtain that

$$\begin{aligned} &\|w_{n+1} - z\|_\infty \\ =\ & \max_{g\in[g_0-\tau,b]} |Hp_n(g) - Hz(g)| \qquad (5.19) \\ =\ & \max_{g\in[g_0-\tau,e]} \Big|\zeta(g_o) \\ &+ \int_{g_0}^{g} f(s, p_n(s), p_n(s-\tau))ds - \zeta(g_o) - \int_{g_0}^{g} f(s, z(s), z(s-\tau))ds\Big| \\ =\ & \max_{g\in[g_0-\tau,e]} \left|\int_{g_0}^{d} f(s, p_n(s), p_n(s-\tau))ds - \int_{g_0}^{g} f(s, z(s), z(s-\tau))ds\right| \\ \leq\ & \max_{g\in[g_0-\tau,e]} \int_{g_0}^{g} |f(s, p_n(s), p_n(s-\tau)) - \int_{g_0}^{g} f(s, z(s), z(s-\tau))|ds \\ \leq\ & \max_{g\in[g_0-\tau,e]} \int_{g_0}^{g} L_f(|p_n(s) - z(s)| + |p_n(s-\tau) - z(s-\tau)|)ds \\ \leq\ & \int_{g_0}^{g} L_f(\max_{g\in[g_0-\tau,e]} |p_n(s) - z(s)| + \max_{g\in[g_0-\tau,e]} |p_n(s-\tau) - q(s-\tau)|)ds \\ \leq\ & \int_{d_0}^{d} L_f(\|p_n - z\|_\infty + \|p_n - z\|_\infty)ds \\ \leq\ & 2L_f(g-g_0)\|p_n - z\|_\infty. \qquad (5.20) \end{aligned}$$

Substituting (5.18) into (5.20) and using condition (T_5), we obtain

$$\begin{aligned} \|w_{n+1} - z\|_\infty &\leq 2L_f(e-g_0)[1 - (\eta_n + \mu_n)(1 - 2L_f(e-g_0))]\|w_n - z\|_\infty \\ &\leq [1 - (\eta_n + \mu_n)(1 - 2L_f(e-g_0))]\|\eta_n - q\|_\infty. \end{aligned} \tag{5.21}$$

Now set $\sigma_n = (\eta_n + \mu_n)(1 - 2L_f(e-g_0)) < 1$. Since $\eta_n + \mu_n \in [0,1]$ and condition (T_5), we know that $\sigma_n \in [0,1]$ such that $\sum_{n=0}^{\infty} \phi_n = \infty$ and set $\theta_n = \|w_n - z\|_\infty$. Then (5.21) now takes the form

$$\theta_{n+1} \leq (1 - \sigma_n)\theta_n. \tag{5.22}$$

Thus all the conditions of Lemma 2.2 are satisfied. Hence, $\lim_{n\to\infty} \|w_n - z\|_\infty = 0$. This completes the proof of Theorem 5.2.

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