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HIGHER ORDER GENERAL CONVEX FUNCTIONS AND GENERAL VARIATIONAL INEQUALITIES

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Abstract. In this paper, we define and consider some new concepts of the higher order general strongly convex functions with respect to an arbitrary function. Some properties of the higher order general strongly convex functions are investigated under suitable conditions. It is shown that the parallelogram laws for Banach spaces can be obtained as applications of higher order general strongly affine convex functions. It is shown that the optimality conditions of the higher order general strongly convex functions are characterized by a class of variational inequalities, which is called the higher order strongly general variational inequality. Auxiliary principle technique is used to suggest an implicit method for solving higher order strongly variational inequalities. Convergence analysis of the proposed method is investigated. Some special cases also discussed. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

1. Introduction

It is well known that the convex sets and convex functions had played crucial important part in the developments of the pure and applied sciences and are continue to inspire novel and innovative applications. Convex functions have been extended and generalized in various directions in recent years. Mohsen et al [17] and Noor and Noor [25] introduced the concept of higher order strongly convex functions and studied their properties. These results can be viewed as a significant refinement of the results of Lin and Fukushima [14] and Alabdali et al [2]. Higher order strongly convex functions include the strongly convex functions, which were introduced and studied by Polyak [30]. Karmardian [11] used the strongly convex functions to discuss the unique existence of a solution of the nonlinear complementarity problems. Awan et a [4, 5] have derived Hermite- Hadamard type

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inequalities for various classes of strongly convex f unctions, which provide upper and lower estimate for the integrand. For the applications of strongly convex functions in optimization, variational inequalities and other branches of pure and applied sciences, see [1–6, 9, 11, 12, 14–17, 19–21, 23–31, 33, 35] and the references therein.

Variational principles theory is a branch of mathematical sciences with a wide range of applications in industry, physical, social, regional and engineering sciences. Researches in this theory have shown important and novel connections with all areas of pure and applied sciences. By variational principles, we mean maximum and minimum problems arising in game theory, mechanics, geometrical optics, general relativity theory, economics, transportation, differential geometry and related areas. Variational inequalities theory with their applications to mathematical physics, pure and applied sciences, was introduced by Stampacchia [32]in 1964. Theory of variational inequalities provides us with a simple, natural, efficient and unified frame work to study a wide class of unrelated problems. It is important to mention that the minimum of a differentiable convex functions on the convex sets can be characterized by variational inequalities. It is amazing that this simple fact has played a significant role in the developments of several areas of pure and applied science. It is known that a set may not be convex set. However, a set can be made convex set with respect to some arbitrary functions. Motivated by this fact, Youness [34] introduced the concept of general convex sets and general convex functions involving an arbitrary function. For different suitable and appropriate choice of the arbitrary function, general convex sets and general convex functions include the general convex sets introduced by Noor [21]as special cases. Noor [21] has shown that the optimality conditions of the differentiable general convex functions on the general convex sets can be characterized a class of variational inequalities called general variational inequality, the origin of which can be traced back to Noor [18]. General variational inequalities can be viewed as a novel and significant of the variational inequalities, which were introduced and studied by Stampacchia cite33. For the formulation, applications, numerical methods,sensitivity analysis and other aspects of variational inequalities, see [9–13, 17–20, 24, 28, 32, 35] and the references therein.

Inspired and motivated by the applications of higher order convex functions in various branches of sciences, we introduce and consider higher order general strongly convex functions with respect to an arbitrary function, which is the main motivation of this paper. Results in this paper can be viewed as a continuation of our previous investigations [25]. Several new concepts of monotonicity are introduced. We establish the relationship between these classes and derive some new results under some mild conditions. It is shown that parallelograms laws for uniform Banach spaces can be obtained from these definitions. In Section 4, we consider higher order general strongly variational inequalities. We have shown that the minimum of a higher order general strongly convex functions on the general convex sets can be characterized by higher order general strongly variational inequalities. Due to the inherent nonlinearity, the projection method and its variant form can not used to suggest the iterative methods for solving these nonlinear variational. To overcome these drawbacks, we use the technique of the auxiliary principle [10] to suggest an implicit method for solving general variational inequalities. Convergence analysis of the proposed method is investigated using the concept of pseudomonotoncity, which is a weaker condition. Some new special cases are discussed, which can be viewed itself an elegant and interesting applications of the higher order strongly general convex functions. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

2. Formulations and basic facts

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively.

Definition 2.1. *[9, 15]* A set K in H is said to be a convex set, if

$$u + t(v - u) \in K, \qquad \forall u, v \in K, t \in [0, 1].$$

Definition 2.2. A function F is said to be convex function, if

$$F((1 - t)u + tv) = (1 - t)F(u) + tF(v); \forall u, v \in K, t \in [0; 1]. \tag{2.1}$$

It is well known that $u \in K$ of a differential convex functions F is equivalent to finding $u \in K$ such that

$$\langle F'(u(, v - u\rangle \geq 0, \forall v \in K, \tag{2.2}$$

which is called the variational inequality, introduced and studied by Stampacchia [32]. Variational inequalities can be regarded as a novel and significant extension of variational principles, the origin of which can be traced back to Euler, Lagrange, Newton and Bernoulli brothers.
We would like to mention that the underlying the set may not be a convex set in many important applications. To overcome this drawback, the set can be made convex set with respect to an arbitrary function, which is called generalized convex set.

Definition 2.3. [34]. The set K_g in H is said to be a general convex set, if there exists an arbitrary function g such that

$$(1 - t)g(u) + tg(v) \in K_g, \quad \forall u, v \in H : g(u), g(v) \in K_g, t \in [0; 1]. \tag{2.3}$$

We now discuss some special cases of the general convex sets.

If $g = I$, the identity operator, then general convex set reduces to the classical convex set. Clearly every convex set is a general convex set, but the converse is not true.

For the sake of simplicity, we always assume that $\forall u, v \in H : g(u), g(v) \in K_g$, unless otherwise.

Definition 2.4. A function F is said to be generalized convex function, if there exists an arbitrary non-negative functions g such that

$$F((1 - t)g(u) + tg(v)) \leq (1 - t)F(g(u)) + tF(g(v)), \quad \forall g(u), g(v) \in K_g, t \in [0, 1]. \tag{2.4}$$

Noor [21] proved that the minimum $u \in H : g(u) \in K_g$ of the differentiable general convex functions F can be characterized by the class of variational inequalities of the type:

$$\langle F'(g(u)), g(v) - g(u)\rangle \geq 0, \quad \forall v \in H : g(v) \in K_g, \tag{2.5}$$

which is known as the general variational inequalities. For the applications of the general variational inequalities in various branches of pure and applied sciences, see [18–27] and the references therein.

We now introduce some new classes of higher order strongly general convex functions and higher order strongly affine general convex functions.

Definition 2.5. A function F on the convex set K_g is said to be higher order strongly general convex with respect to an arbitrary function g, if there exists a constant $\mu > 0$, such that

$$\begin{aligned} F(g(u) + t(g(v) - g(u))) \quad &\leq \quad (1-t)F(g(u)) + tF(g(v)) \\ &\quad -\mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - g(u)\|^p, \\ &\quad \forall g(u), g(v) \in K_g, t \in [0,1], p > 1. \end{aligned} \tag{2.6}$$

A function F is said to higher order strongly general concave, if and only if, $-F$ is higher order strongly general convex.
If $t = \frac{1}{2}$, then

$$F\left(\frac{g(u)+g(v)}{2}\right) \leq \frac{F(g(u)) + F(g(v))}{2} - \mu\frac{1}{2^p}\|g(v) - g(u)\|^p, \qquad \forall g(u), g(v) \in K_g, t \in [0,1], p > 1.$$

The function F is said to be higher order strongly general J-convex function.
We now discuss some special cases.

I. If $g = I$, then definition 2.5 reduces to

Definition 2.6. A function F on the convex set K is said to be higher order strongly convex function, if there exists a constant $\mu > 0$, such that

$$F(u + t(v-u)) \leq (1-t)F(u) + tF(v) - \mu\{t^p(1-t) + t(1-t)^p\}\|v - u\|^p, \qquad \forall v \in K, t \in [0,1], p > 1.$$

which were introduced and studied by Noor and Noor [25] and Mohsen et al [17]. It have been shown that the higher order strongly convex functions [17, 25] can be viewed as important and significant refinement of the previously known results.

II. If $p = 2$, then the order strongly general convex function becomes strongly general convex functions, that is,

$$F(g(u) + t(g(v(-g(u))) \leq (1-t)F(g(u)) + tF(g(v)) - \mu t(1-t)\|g(v) - g(u)\|^2, \qquad \forall g(u), g(v) \in K_g, t \in [0,1].$$

For the properties of the strongly general convex functions in variational inequalities and equilibrium problems, see Noor [17, 25].

Definition 2.7. A function F on the convex set K is said to be higher order strongly general quasi-convex, if there exists a constant $\mu > 0$ such that

$$\begin{aligned} F(g(u)+t(g(v)-g(u)) &\leq \max\{F(g(u)),F(g(v))\} \\ &\quad -\mu\{t^p(1-t)+t(1-t)^p\}\|g(v)-g(u)\|^p, \\ &\quad \forall g(u),g(v)\in K_g, t\in[0,1], p>1. \end{aligned}$$

Definition 2.8. A function F on the convex set K is said to be higher order strongly general log-convex, if there exists a constant $\mu > 0$ such that

$$\begin{aligned} F(g(u)+t(g(v)-g(u)) &\leq (F(g(u)))^{1-t}(F(g(v)))^t \\ &\quad -\mu\{t^p(1-t)+t(1-t)^p\}\|g(v)-g(u)\|^p, \\ &\quad \forall g(u),g(v)\in K_g, t\in[0,1], P>1, \end{aligned}$$

where $F(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned} & F(g(u)+t(g(v)-g(u)) \\ \leq\ & (F(g(u)))^{1-t}(F(g(v)))^t-\mu\{t^p(1-t)+t(1-t)^p\}\|g(v)-g(u)\|^p \\ \leq\ & (1-t)F(g(u))+tF(g(v))-\mu\{t^p(1-t)+t(1-t)^p\}\|g(v)-g(u)\|^p \\ \leq\ & \max\{F(g(u)),F(g(v))\}-\mu\{t^p(1-t)+t(1-t)^p\}\|g(v)-g(u)\|^p. \end{aligned}$$

This shows that every higher order strongly general log-convex function is a higher order strongly general convex function and every higher order strongly general convex function is a higher order strongly general quasi-convex function. However, the converse is not true.

Definition 2.9. A function F on the convex set K_g is said to be a higher order strongly affine function, if there exists a constant $\mu > 0$, such that

$$\begin{aligned} F(g(u)+t(g(v)-g(u)) &= (1-t)F(g(u))+tF(g(v)) \\ &\quad -\mu\{t^p(1-t)+t(1-t)^p\}\|g(v)-g(u)\|^p, \\ &\quad \forall g(u),g(v)\in K_g, t\in[0,1]. \end{aligned}$$

Note that if a functions is both higher order strongly general convex and higher order strongly general concave, then it is higher order strongly affine general convex function.

Definition 2.10. A function F is called higher order strongly general quadratic equation, if there exists a constant $\mu > 0$, such that

$$\begin{aligned} F\left(\frac{g(u)+g(v)}{2}\right) &= \frac{F(g(u))+F(g(v))}{2}-\mu\frac{1}{2^p}\|g(v)-g(u)\|^2, \\ &\quad \forall g(u),g(v)\in K_g, t\in[0,1]. \end{aligned}$$

This function F is also called higher order strongly affine general J-convex function.

For appropriate and suitable choice of the arbitrary function g, the parameter μ and p, one can obtain several new and known classes of strongly convex functions and their variant forms as special cases of higher order strongly general convex functions. This shows that the class of higher order strongly general convex functions is quite broad and unifying one.

Definition 2.11. An operator $T : K \to H$ is said to be:

(1) higher order strongly monotone, if and only if, there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, g(u) - g(v)\rangle \geq \alpha \|g(v) - g(u)\|^p, \quad \forall g(u), g(v) \in K_g.$$

(2) higher order strongly pseudomonotone, if and only if, there exists a constant $\nu > 0$ such that

$$\begin{aligned}&\langle Tu, g(v) - g(u)\rangle + \nu\|g(v) - g(u)\|^p \geq 0\\ &\Rightarrow\\ &\langle Tv, g(v) - g(u)\rangle \geq 0, \quad \forall g(u), g(v) \in K_g.\end{aligned}$$

(3) higher order strongly relaxed pseudomonotone, if and only if, there exists a constant $\mu > 0$ such that

$$\begin{aligned}&\langle Tu, g(v) - g(u)\rangle \geq 0\\ &\Rightarrow\\ &-\langle Tv, g(u) - g(v)\rangle + \mu\|v - u\|^p \geq 0, \quad \forall g(u), g(v) \in K_g.\end{aligned}$$

Definition 2.12. A differentiable function F on the convex set K_g is said to be higher order strongly general pseudo-convex function, if and only if, if there exists a constant $\mu > 0$ such that

$$\langle F'(u), g(v) - g(u)\rangle + \mu\|g(v) - g(u)\|^p \geq 0 \Rightarrow F(g(v)) \geq F(g(u)), \quad \forall g(u), g(v) \in K_g.$$

3. Main results

In this section, we discuss some basic properties of higher order strongly general convex functions.

Theorem 3.1. *Let F be a differentiable function on the general convex set K_g. Then the function F is higher order strongly general convex function, if and only if,*

$$\begin{aligned}F(g(v)) - F(g(u)) \quad &\geq \quad \langle F'(g(u)), g(v) - g(u)\rangle + \mu\|g(v) - g(u)\|^p, \\ &\quad\ \forall g(v), g(u) \in K_g, t \in [0, 1].\end{aligned} \tag{3.1}$$

Proof. Let F be a higher order strongly generalized convex function on the general convex set K_g. Then

$$\begin{aligned}F(g(u) + t(g(v) - g(u))) &\leq (1 - t)F(g(u)) + tF(g(v))\\ &\quad -\mu\{t^p(1 - t) + t(1 - t)^p\}\|g(v) - g(u)\|^p,\\ &\quad \forall g(v), g(u) \in K_g, t \in [0.1].\end{aligned}$$

which can be written as

$$\begin{aligned}F(g(v)) - F(g(u)) \geq &\frac{F(g(u) + t(g(v) - g(u)) - F(g(u))}{t}\\ &+\{t^{p-1}(1 - t) + (1 - t)^p\}\|g(v) - g(u)\|^p.\end{aligned}$$

Taking the limit in the above inequality as $t \to 0$,, we have

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), g(v) - g(u))\rangle + \mu\|g(v) - g(u)\|^p, \forall g(v), g(u) \in K_g.$$

which is (3.1), the required result.

Conversely, let (3.1) hold. Then, $\forall g(v), g(u) \in K_g, t \in [0,1]$, $g(v_t) = g(u) + t(g(v) - g(u)) \in K_g$, we have

$$\begin{aligned} F(g(v)) - F(g(v_t)) &\geq \langle F'(g(v_t)), g(v) - g(v_t))\rangle + \mu\|g(v) - g(v_t)\|^p \\ &= (1-t)\langle F'(g(v_t)), g(v) - g(u)\rangle \\ &\quad +\mu(1-t)^p\|g(v) - g(u)\|^p, \forall g(v), g(u) \in K_g. \end{aligned} \tag{3.2}$$

In a similar way, we have

$$\begin{aligned} F(g(u)) - F(g(v_t)) &\geq \langle F'(g(v_t)), g(u) - g(v_t))\rangle + \mu\|u - v_t\|^p \\ &= -t'langleF'(g(v_t)), g(v) - g(u)\rangle + \mu t^p\|g(v) - g(u)\|^p. \end{aligned} \tag{3.3}$$

Multiplying (3.2) by t and (3.3) by $(1-t)$ and adding the resultant, we have

$$\begin{aligned} F(g(u) + t(g(v) - g(u)) &\leq (1-t)F(g(u)) + tF(g(v)) \\ &- \mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - g(u)\|^p, \forall g(v), g(u) \in K_g, \end{aligned}$$

showing that F is a higher order strongly general convex function. □

Theorem 3.2. *Let F be a differentiable higher order strongly general convex function on the convex set K_g. Then $F'(.)$ is a higher order strongly monotone operator.*

Proof. Let F be a higher order strongly general convex function on the convex set K_g. Then, from Theorem 3.1, we have

$$\begin{aligned} F(g(v)) - F(g(u)) &\geq \langle F'(g(u)), g(v) - g(u)\rangle + \mu\|g(v) - g(u)\|^p, \\ &\qquad \forall g(v), g(u) \in K_g. \end{aligned} \tag{3.4}$$

Changing the role of $g(u)$ and $g(v)$ in (3.4), we have

$$\begin{aligned} F(g(u)) - F(g(v)) &\geq \langle F'(g(v)), g(u) - g(v))\rangle + \mu\|g(v) - g(u)\|^p, \\ &\qquad \forall g(v), g(u) \in K_g. \end{aligned} \tag{3.5}$$

Adding (3.4) and (3.5), we have

$$\langle F'(g(u)) - F'(g(v)), g(u) - g(v)\rangle \geq 2\mu\|g(v) - g(u)\|^p, \quad \forall g(v), g(u) \in K_g. \tag{3.6}$$

which shows that $F'(.)$ is a higher order strongly monotone operator.

□

We remark that the converse of Theorem 3.2 is not true. However, we have the following result.

Theorem 3.3. *If the differential operator $F'(.)$ of a differentiable higher order strongly general convex function F is higher order strongly monotone operator, then*

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), g(v) - g(u)\rangle + 2\mu\frac{1}{p}\|g(v) - g(u)\|^p, \forall g(v), g(u) \in K_g \tag{3.7}$$

Proof. Let $F'(.)$ be a higher order strongly monotone operator. Then, from (3.6), we have

$$\begin{aligned} \langle F'(g(v)), g(u) - g(v)\rangle &\geq \langle F'(g(u)), g(u) - g(v))\rangle + 2\mu\|g(v) - g(u)\|^p, \\ &\qquad \forall g(v), g(u) \in K_g. \end{aligned} \tag{3.8}$$

Since K is an convex set, $\forall g(v), g(u) \in K_g$, $t \in [0,1]$, $g(v_t) = g(u) + t(g(v) - g(u)) \in K_g$. Taking $g(v) = g(v_t)$ in (3.8), we have

$$\begin{aligned} \langle F'(g(v_t)), g(u) - g(v_t)\rangle &\leq -\langle F'(g(u)), g(u) - g(v_t)\rangle - 2\mu\|g(v) - g(u)\|^p \\ &= -t\langle F'(g(u)), g(v) - g(u)\rangle - 2\mu t^p\|g(v) - g(u)\|^p, \end{aligned}$$

which implies that

$$\langle F'(g(v_t)), g(v) - g(u)\rangle \geq \langle F'(g(u)), g(v) - g(u)\rangle + 2\mu t^{p-1}\|g(v) - g(u)\|^P. \quad (3.9)$$

Consider the an arbitrary auxiliary function

$$\xi(t) = F(g(u) + t(g(v) - g(u)), \forall g(v), g(u) \in K_g,$$

from which, we have

$$\xi(1) = F(g(v)), \quad \xi(0) = F(g(u)).$$

Then, from (3.9), we have

$$\begin{aligned}\xi'(t) &= \langle F'(g(v_t), g(v) - g(u)\rangle \\ &\geq \langle F'(g(u)), g(v) - g(u)\rangle + 2\mu t^{p-1}\|g(v) - g(u)\|^p. \end{aligned} \quad (3.10)$$

Integrating (3.10) between 0 and 1, we have

$$\xi(1) - \xi(0) = \int_0^1 \xi'(t)dt \geq \langle F'(g(u)), g(v) - g(u)\rangle + 2\mu\frac{1}{p}\|g(v) - g(u)\|^p.$$

Thus it follows that

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), g(v) - g(u)\rangle + 2\mu\frac{1}{p}\|g(v) - g(u)\|^p, \forall g(v), g(u) \in K_g,$$

which is the required (3.7). □

We note that, if $p = 2$, then Theorem 3.3 can be viewed as the converse of Theorem 3.2.

We now give a necessary condition for higher order strongly generalized pseudo-convex function.

Theorem 3.4. *Let $F'(.)$ be a higher order strongly relaxed pseudomonotone operator. Then F is a higher order strongly general pseudo-convex function.*

Proof. Let F' be a higher order strongly relaxed pseudomonotone operator. Then,

$$\langle F'(g(u)), g(v) - g(u)\rangle \geq 0, \forall g(u), g(v) \in K_g,$$

implies that

$$\langle F'(g(v)), g(v) - g(u)\rangle \geq \mu\|g(v) - g(u)\|^p, \forall g(u), g(v) \in K_g. \quad (3.11)$$

Since K_g is a general convex set, $\forall g(u), g(v) \in K_g, \quad t \in [0, 1]$, $g(v_t) = g(u) + t(g(v) - g(u)) \in K_g$.
Taking $g(v) = g(v_t)$ in (3.11), we have

$$\langle F'(g(v_t), g(v) - g(u)\rangle \geq \mu t^{p-1}\|g(v) - g(u)\|^p. \quad (3.12)$$

Consider the an arbitrary auxiliary function

$$\xi(t) = F(g(u) + t(g(v) - g(u))) = F(g(v_t)), \quad \forall g(u), g(v) \in K_g, t \in [0, 1],$$

which is differentiable, since F is differentiable function. Then, using (3.12), we have

$$\xi'(t) = \langle F'(g(v_t)), g(v) - g(u))\rangle \geq \mu t^{p-1}\|g(v) - g(u)\|^p.$$

Integrating the above-mentioned relation between 0 to 1, we have

$$\xi(1) - \xi(0) = \int_0^1 \xi'(t)dt \geq \frac{\mu}{p}\|g(v) - g(u)\|^p,$$

that is,

$$F(g(v)) - F(g(u)) \geq \frac{\mu}{p}\|g(v) - g(u)\|^p), \forall g(u), g(v) \in K_g,$$

showing that F is a higher order strongly general pseudo-convex function. □

Definition 3.5. A function F is said to be sharply higher order general strongly pseudo-convex, if there exists a constant $\mu > 0$ such that

$$\begin{aligned} &\langle F'(g(u)), g(v) - g(u)\rangle \geq 0 \\ &\Rightarrow \\ &F(g(v)) \geq F(g(v) + t(g(u) - g(v))) \\ &+\mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - g(u)\|^p, \forall g(u), g(v) \in K_g. \end{aligned}$$

Theorem 3.6. *Let F be a sharply higher order general strongly pseudo-convex function on the general K_g with a constant $\mu > 0$. Then*

$$\langle F'(g(v)), g(v) - g(u)\rangle \geq \mu\|g(v) - g(u)\|^p, \forall g(u), g(v) \in K_g.$$

Proof. Let F be a sharply higher order general strongly pesudo-convex function on the general K_g. Then

$$F(g(v)) \geq F(g(v) + t(g(u) - g(v))) + \mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - g(u)\|^p, \quad \forall g(u), g(v) \in K_g, t \in [0, 1],$$

from which, we have

$$\frac{F(g(v) + t(g(u) - g(v)) - F(g(v))}{t} + \mu\{t^{p-1}(1-t) + (1-t)^p\}\|g(v) - g(u)\|^p \geq 0.$$

Taking limit in the above inequality, as $t \to 0$, we have

$$\langle F'(g(v)), g(v) - g(u)\rangle \geq \mu\|g(v) - g(u)\|^p, \forall g(u), g(v) \in K_g,$$

the required result. □

Theorem 3.7. *Let f be a higher order general strongly affine convex function. Then F is a higher order general strongly convex function, if and only if, $H = F - f$ is a general convex function.*

Proof. Let f be a higher order general strongly affine convex function, Then

$$\begin{aligned} f((1-t)g(u) + tg(v)) \quad = \quad & (1-t)f(g(u)) + tf(g(v)) \\ & -\mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - g(u)\|^p, \\ & \forall g(u), g(v) \in K_g. \end{aligned} \tag{3.13}$$

From the higher order general strongly convexity of F, we have

$$\begin{aligned} F((1-t)g(u) + tg(v)) \quad \leq \quad & (1-t)F(g(u)) + tF(g(v)) \\ & -\mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - g(u)\|^p, \\ & \forall g(u), g(v) \in K_g. \end{aligned} \tag{3.14}$$

From (3.13) and (3.14), we have

$$\begin{aligned} & F((1-t)g(u) + tg(v)) - f((1-t)f(g(u)) + tf(g(v)) \\ & \leq (1-t)(F(g(u)) - f(g(u))) + t(F(g(v)) - f(g(v))), \end{aligned} \tag{3.15}$$

from which it follows that

$$\begin{aligned} H((1-t)g(u)+tg(v)) &= F((1-t)g(u)+tg(v)) - f((1-t)g(u)+tg(v)) \\ &\leq (1-t)F(g(u))+tF(g(v)) - (1-t)f(g(u)(-tf(g(v)) \\ &= (1-t)(F(g(u))-f(g(u)))+t(F(g(v))-f(g(v))), \end{aligned}$$

which show that $H = F - f$ is a general convex function.
The inverse implication is obvious. □

Definition 3.8. A function F is said to be a pseudo-convex function with respect to a strictly positive bifunction $B(.,.)$, if

$$\begin{aligned} &F(g(v)) < F(g(u)) \\ &\Rightarrow \\ &F(g(u)+(1-t)(g(v),g(u))) < F(g(u))+t(t-1)B(g(v),g(u)), \\ &\forall (u), g(v) \in K_g, t \in [0,1]. \end{aligned}$$

Theorem 3.9. *If the function F is higher order geeneral strongly convex function such that $F(g(v)) < F(g(u))$, then the function F is higher order general strongly pseudo-convex function*

Proof. Since $F(g(v)) < F(g(u))$ and F is a higher order general strongly convex function, then
$\forall g(u), g(v) \in K_g, \quad t \in [0,1]$, we have

$$\begin{aligned} & F(g(u)+t(g(v)-g(u))) \leq F(g(u))+t(F(g(v))-F(g(u))) \\ & -\mu\{t^p(1-t)+t(1-t)^p\}\|g(v)-g(u)\|^p \\ < \ & F(g(u))+t(1-t)(F(g(v))-F(g(u))) - \mu\{t^p(1-t)+t(1-t)^p\}\|g(v)-g(u)\|^p \\ = \ & F(g(u))+t(t-1)(F(g(u))-F(g(v))) - \mu\{t^p(1-t)+t(1-t)^p\}\|g(v)-g(u)\|^p \\ < \ & F(g(u))+t(t-1)B(g(u),g(v)) - \mu\{t^p(1-t)+t(1-t)^p\}\|g(v)-g(u)\|^p, \\ & \forall g(u), g(v) \in K_g, \end{aligned}$$

where $B(g(u),g(v)) = F(g(u)) - F(g(v)) > 0$. Hence the function F is a higher order general strongly convex function, the required result. □

We now show that uniformly Banach spaces can be characterized by the parallelogram laws, which can be obtained from the higher order general strongly affine convexity.
Setting $F(u) = \|u\|^p$ in Definition 2.9, we have

$$\begin{aligned} \|g(u)+t(g(v)-g(u))\|^p &= (1-t)\|g(u)\|^p + t\|g(v)\|^p \\ & \quad -\mu\{t^p(1-t)+t(1-t)^p\}\|g(v)-g(u)\|^p, \quad (3.16) \\ & \quad \forall g(u), g(v) \in K_g, t \in [0,1]. \end{aligned}$$

Taking $t = \frac{1}{2}$ in (3.16), we have

$$\|\frac{g(u)+g(v)}{2}\|^p + \mu\frac{1}{2^p}\|g(v)-g(u)\|^p = \frac{1}{2}\|g(u)\|^p + \frac{1}{2}\|g(v)\|^p, \forall g(u), g(v) \in K_g, \quad (3.17)$$

which implies that

$$\|g(u)+g(v)\|^p + \mu\|g(v)-g(u)\|^p = 2^{p-1}\{\|g(u)\|^p + \|g(v)\|^p\}, \forall g(u), g(v) \in K_g, \quad (3.18)$$

which is known as the general parallelogram for the l^p-spaces and can be viewed as a novel applications of the higher order general strongly affine convex functions. The equation (3.18) can be used to characterize the uniformly Banach spaces involving an arbitrary function, which is itself an interesting applications of the parallelogram laws. For $p = 2$, the general parallelogram law (3.18) reduces to

$$\|g(u) + g(v)\|^2 + \mu\|g(v) - g(u)\|^2 = 2\{\|g(u)\|^2 + \|g(v)\|^2\}, \forall g(u), g(v) \in K_g,$$

which is a new characterization of the inner produce spaces involving an arbitrary function.

For $g = I$, the identity operator, Xi [33] obtained these characteristics of p-uniform convexity and q-uniform smoothness of a Banach space via the functionals $\|.\|^p$ and $\|.\|^q$, respectively. Bynum [6] and Chen et al [7, 8] have studied the properties and applications of the parallelogram laws for the Banach spaces.

4. General variational inequalities

In this section, we consider a higher order general strongly variational inequality problem.
To be more precise, for given two operators T, g, we consider the problem of finding $u \in K$ for a constant μ such that

$$\langle Tu, g(v) - g(u)\rangle + \mu\|g(v) - g(u)\|^p \geq 0, \forall g(v) \in K, p > 1, \tag{4.1}$$

which is called the higher order general strongly variational inequality.

(I). If $\mu = 0$, then (4.1) is equivalent to finding $u \in K$, such that

$$\langle Tu, g(v) - g(u)\rangle \geq 0, \forall g(v) \in K, \tag{4.2}$$

which is known as the general variational inequality, which was introduced and studied by Noor [18] in 1988. For recent developments in general variational inequalities, see Noor [19–23]and the references therein.

(II). If $p = 1$, then problem (4.1) reduces to: For given operators T, g, we consider the problem of finding $u \in K$ for a constant μ such that

$$\langle Tu, g(v) - g(u)\rangle + \mu\|g(v) - g(u)\| \geq 0, \forall g(v) \in K, p > 1, \tag{4.3}$$

which is called the approximate general variational inequality.

(III). If $p = 2$, then (4.1) is equivalent to finding $u \in K$, such that

$$\langle Tu, g(v) - g(u)\rangle + \mu\|g(v) - g(u)\|^2 \geq 0, \forall g(v) \in K,$$

which is called the general strongly variational inequality.

For suitable and appropriate choice of arbitrary functions T, g, the parameter μ and p, one can obtain several new and known classes of variational inequalities, see [12, 14, 19–24, 27].

We now show that the higher order strongly general variational inequalities (4.1) arise as the optimality criteria of a differentiable general strongly convex functions, which is the main motivation of our next result.

Theorem 4.1. *Let F be a differentiable higher order strongly general convex function with modulus $\mu > 0$. If $u \in H : g(u) \in K_g$ is the minimum of the function F, then*

$$F(g(v)) - F(g(u)) \geq \mu \|g(v) - g(u)\|^p, \quad \forall g(u), g(v) \in K_g. \tag{4.4}$$

Proof. Let $u \in H : g(u) \in K_g$ be a minimum of the function F. Then

$$F(g(u)) \leq F(g(v)), \forall g(v) \in K_g. \tag{4.5}$$

Since K_g is a general convex set, so, $\forall g(u), g(v) \in K_g, \quad t \in [0,1]$,

$$g(v_t) = (1-t)g(u) + tg(v) \in K_g.$$

Taking $g(v) = g(v_t)$ in (4.5), we have

$$0 \leq \lim_{t \to 0}\{\frac{F(g(u) + t(g(v) - g(u))) - F(g(u))}{t}\} = \langle F'(g(u)), g(v) - g(u)\rangle. \tag{4.6}$$

Since F is a differentiable higher order general strongly convex function, so

$$\begin{aligned} F(g(u) + t(g(v) - g(u))) \quad &\leq \quad F(g(u)) + t(F(g(v)) - F(g(u))) \\ &\quad -\mu\{t^p(1-t) + t(1-t)^p\}\|g(v) - g(u)\|^p, \forall g(u), g(v) \in K_g, \end{aligned}$$

from which, using (4.6), we have

$$\begin{aligned} F(g(v)) - F(g(u)) \quad &\geq \quad \lim_{t \to 0} \frac{F(g(u) + t(g(v) - g(u))) - F(g(u))}{t} + \mu\|g(v) - g(u)\|^p \\ &= \quad \langle F'(g(u)), g(v) - g(u)\rangle + \mu\|g(v) - g(u)\|^p, \end{aligned}$$

the required result (4.4). □

Clearly problem (4.4) is exactly the problem (4.1) with $Tu = F'(g(u))$.

We now consider some iterative methods for solving the problem (4.1). We remark that the projection method and its variant forms can be used to study the higher order general strongly variational inequalities (4.1)due to its inherent structure. To overcome this drawback, we consider the auxiliary principle technique, which is mainly due to Glowinski et al [10] and Lions and Stampacchia [13] as developed by Noor [20]. We use this technique to suggest some iterative methods for solving the general variational inequalities (4.1).

For given $u \in H : g(u) \in K$ satisfying (4.1), consider the problem of finding $w \in H : g(w) \in K$, such that

$$\begin{aligned} \langle \rho Tw, g(v) - g(w)\rangle + \langle w - u + \alpha(u - u), v - w\rangle + \nu\|g(v) - g(w)\|^p \geq 0, \\ \forall g(v) \in K, p > 1, \end{aligned} \tag{4.7}$$

where $\rho > 0, \quad \alpha$ are parameters. The problem (4.7) is called the auxiliary higher order general strongly variational inequality. It is clear that the relation (4.7) defines a mapping connecting the problems (4.1) and (4.7). We not that, if $w(u) = u$, then w is a solution of problem (4.1). This simple observation enables to suggest an iterative method for solving (4.1).

Algorithm 4.2. . For given $u_0 \in K$, find the approximate solution u_{n+1} by the scheme

$$\begin{aligned} \langle \rho Tu_{n+1}, g(v) - g(u_{n+1})\rangle + \langle u_{n+1} - u_n + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \\ +\nu\|g(v) - g(u_{n+1})\|^p \geq 0. \forall g(v) \in K, p > 1. \end{aligned} \tag{4.8}$$

The Algorithm 4.2 is known as the implicit method. Such type of methods have been studied extensively for various classes of variational inequalities. See [12, 13] and the reference therein. If $\nu = 0$, then Algorithm 4.2 reduces to:

Algorithm 4.3. For given $u_0 \in K$, find the approximate solution u_{n+1} by the scheme
$$\langle \rho T u_{n+1}, g(v) - g(u_{n+1})\rangle + \langle u_{n+1} - u_n + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \forall g(v) \in K,$$
which appears to be new ones even for solving the general variational inequalities.

In order to study the convergence analysis of Algorithm 4.2, we need the following.

Definition 4.4. The operator T is said to be pseudo g-monotone with respect to $\mu\|g(v) - g(u)\|^p, p > 1$, if
$$\langle \rho T u, g(v) - g(u)\rangle + \mu\|g(v) - g(u)\|^p \geq 0, \forall g(v) \in K, p > 1,$$
$$\Longrightarrow$$
$$\langle \rho T v, g(v) - g(u)\rangle - \mu\|g(u) - g(v)\|^p \geq 0, \forall g(v) \in K, p > 1$$

If $\mu = 0$, then Definition 4.4 reduces to:

Definition 4.5. The operator T is said to be pseudo g-monotone, if
$$\langle \rho T u, g(v) - g(u)\rangle \geq 0, \forall g(v) \in K$$
$$\Longrightarrow$$
$$\langle \rho T v, g(v) - g(u)\rangle \geq 0, \forall g(v) \in K,$$
which appears to be a new one.

We now study the convergence analysis of Algorithm 4.2 for the case $\alpha = 0$.

Theorem 4.6. *Let $u \in K$ be a solution of (4.1) and u_{n+1} be the approximate solution obtained from Algorithm 4.2. If T is a pseudo hg- monotone operator, then*
$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2. \tag{4.9}$$

Proof. Let $u \in H : g(u) \in K$ be a solution of (4.1), then
$$\langle \rho T u, g(v) - g(u)\rangle + \mu\|g(v) - g(u)\|^p, \forall g(v) \in K,$$
implies that
$$\langle \rho T v, g(u) - g(v)\rangle - \mu\|g(u) - g(v)\|^p, \forall g(v) \in K, \tag{4.10}$$
Now taking $v = u_{n+1}$ in (4.10), we have
$$\langle \rho T u_{n+1}, g(u_{n+1}) - g(u)\rangle - \mu\|g(u_{n+1}) - g(u)\|^p \geq 0. \tag{4.11}$$
Taking $v = u$ in (4.8), we have
$$\begin{aligned}\langle \rho T u_{n+1}, g(u) - g(u_{n+1})\rangle + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \\ +\nu\|g(u) - g(u_{n+1})\|^p \geq 0. \quad \forall g(v) \in K, p > 1.\end{aligned} \tag{4.12}$$
Combining (4.11) and (4.12), we have
$$\langle u_{n+1} - u_n, u_{n+1} - u\rangle \geq 0.$$
Using the inequality
$$2\langle a, b\rangle = \|a + b\|^2 - \|a\|^2 - \|b\|^2, \forall a, b \in H,$$
we obtain
$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2,$$
the required result (4.9). □

Theorem 4.7. *Let the operator T be a pseudo g-monotone. If u_{n+1} be the approximate solution*

obtained from Algorithm 4.2 and $u \in H : g(u) \in K$ is the exact solution (4.1), then

$$\lim_{n\to\infty} u_n = u.$$

Proof. Let $u \in H : g(u) \in K$ be a solution of (4.1). Then, from (4.9), it follows that the sequence $\{\|u - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. From (4.9), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - u\|^2,$$

from which, it follows that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{4.13}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence u_n converge to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (4.8), taking the limit $n_j \to 0$ and from (4.13), we have

$$\langle T\hat{u}, g(v) - g(\hat{u})\rangle + \mu\|g(v) - g(\hat{u})\|^p, \quad \forall g(v) \in K, p > 1.$$

This implies that $\hat{u} \in K$ satisfies (4.1) and

$$\|u_{n+1} - u_n\|^2 \leq \|u_n - \hat{u}\|^2.$$

Thus it follows from the above inequality that the sequence u_n has exactly one cluster point $\hat{u}$ and

$$\lim_{n\to\infty} u_n = \hat{u}.$$

□

In order to implement the implicit Algorithm 4.2, one uses the predictor-corrector technique. Consequently, Algorithm 4.2 for the case $\alpha = 0$, is equivalent to the following two-step iterative method for solving the general variational inequality (4.1).

Algorithm 4.8. For a given $u_0 \in K$, find the approximate solution u_{n+1} by the schemes

$$\langle \rho T u_n, g(v) - g(y_n)\rangle + \langle y_n - u_n, v - y_n\rangle + \mu\|g(v) - g(y_n)\|^p \geq 0, \forall g(v) \in K, p > 1$$

$$\langle \rho T y_n, g(u_n) - g(y_n)\rangle + \langle u_n - y_n, v - y_n\rangle + \mu\|g(v) - g(u_n)\|^p \geq 0, \forall g(v) \in K, p > 1.$$

Algorithm 4.8 is called the predictor-corrector iterative method and appears to be a new one.

Using the auxiliary principle technique, we now suggest an other iterative method for solving the general higher order strongly variational inequalities and related optimization problems.

For a given $u \in H : g(u) \in K$ satisfying (4.1), consider the problem of finding $w \in H : h(w) \in K$, such that

$$\langle \rho T u, g(v) - g(w)\rangle + \langle g(w) - g(u), g(v) - g(w)\rangle + \nu\|g(v) - g(w)\|^p \geq 0, \quad \forall g(v) \in K, p > 1, \tag{4.14}$$

where $\rho > 0$ is a parameter. The problem (4.14) is called the auxiliary higher order general strongly variational inequality. It is clear that the relation (4.14) defines a mapping connecting the problems (4.1) and (4.14). We note that, if $w(u) = u$, then w is a solution of problem (4.1). This simple observation enables to suggest an iterative method for solving (4.1).

Algorithm 4.9. For given $u_0 \in K$, find the approximate solution u_{n+1} by the scheme

$$\langle \rho T u_n, g(v) - g(u_{n+1})\rangle + \langle g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle$$
$$+\nu \|g(v) - g(u_{n+1})\|^p \geq 0. \forall g(v) \in K, p > 1, \qquad (4.15)$$

which is an explicit algorithm.

It is worth mentioning that the auxiliary principle technique can be used efficiently to suggest a wide class of iterative methods for solving higher order general strongly variational inequalities. We have only given some glimpses of the higher order general strongly variational inequalities. It is an interesting problem to explore the applications of such type variational inequalities in various fields of pure and applied sciences.

Conclusion

In this paper, we have introduced and studied a new class of convex functions, which is called higher order general strongly convex function. It is shown that several new classes of strongly convex functions can be obtained as special cases of these higher order strongly convex functions. We have studied the basic properties of these functions. We have shown that one can derive the parallelogram laws in Banach spaces, which have applications in prediction theory and stochastic analysis. We have characterized the optimality conditions of higher order general strongly convex functions by a class of variational inequalities, which is called higher order general strongly variational inequality. We have used the auxiliary principle technique to suggest some iterative methods for solving the higher order general strongly variational inequalities. Convergence is also studied under some weak conditions. Several important special cases are also discussed, which are obtained from our results. The interested readers may explore the applications and other properties of the higher order general strongly convex functions in various fields of pure and applied sciences. This is an interesting direction of future research.

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