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THE STRONG AND WEAK SOLUTIONS OF FUZZY OPTIMAL CONTROL PROBLEM WITH STATE CONDITIONS AT THE FINAL TIME

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Abstract. In this paper, with and without restrictions imposed on the fuzzy objective functional, we define two different types of fuzzy Hamiltonian function used together with the concepts of parameterizing the fuzzy valued function and its differentiation and integration by the left and right-hand functions of its $\alpha-$level set and fuzzy variational approaches to prove the necessary conditions for fuzzy optimal control problem with state conditions at the final time. In addition, we introduce the concepts of strong and weak solutions to our problem to guarantee that the optimal solutions are fuzzy functions. Three examples are given to illustrate our main results.

1. Introduction

Classical optimal control theory, despite its modern origins, from a mathematical point of view, it is considered as a variant of one of the oldest and most important subfields in mathematics, the calculus of variations. Meanwhile, it is a powerful mathematical tool that can be used to make decisions in real life, such as, make decisions for controlling diseases and epidemics management. The main purpose of optimal control theory is to maximize the return from or minimize the cost of the operation of physical, social, economic, and biological processes. In this research area, there exists a large literature (seem [1] and the references therein).

On the other hand, many dynamical systems have uncertainty in their input, output, and manner, for that, fuzziness is a good way to express uncertainty for these dynamical systems. Since Zadeh [20] introduced the concepts of fuzzy sets and fuzzy numbers and Chang and Zadeh[4] proposed the concepts of fuzzy mapping and its derivative, a large number of researchers studied many aspects of the

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theory and applications of these notions such as linear programming, optimization, and optimal control problem. In the last years, many authors have been interested in the study of fuzzy optimal control problem and this field attracted a great deal of attention and many results of research have been reported in the literature [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19].

In addition to minimizing (maximizing) terms over the entire time interval, we wish to also minimize (maximize) a function value at one particular point in time, specifically, the end of the time interval [1, 2]. To handle this matter in a fuzzy environment, we will define the fuzzy optimal control problem with state conditions at the final time and then derive the necessary conditions for this problem as the main goal of this paper. In order to achieve our goal, we define two types of fuzzy Hamiltonian function based on the restrictions imposed on the fuzzy objective functional. Solving the necessary conditions is considered as a classical method to provide the optimal solutions to our problem, i.e., optimal fuzzy control and corresponding optimal fuzzy state. So, in order to guarantee that the optimal solutions of the fuzzy optimal control problem with state conditions at the final time are always fuzzy functions, we proposed the concepts of strong and weak solutions to this problem which were first introduced in [18].

The remainder of this paper is organized as follows. In Section 2, we present some basic terminologies and notations that are key to our discussion. Some basic elements of calculus of variations are recalled in Section 3. In Section 4, we establish the main results of this paper in Theorem 4.4 and Theorem 4.7, that provides the necessary conditions for fuzzy optimal control problem with state conditions at the final time and Definition 4.8, that define the concepts of strong and weak solutions of our problem. In Section 5, we discuss the applicability of Theorem 4.4, Theorem 4.7 and Definition 4.8 throughout three examples. In Section 6, we present some concluding remarks.

2. Notations and Preliminaries

In this section, we recall some basic definitions and lemma for a better understanding of this work, which are discussed clearly in [3, 5, 8, 10, 7, 6, 20].

A fuzzy set $\tilde{a} : \mathbb{R} \to [0,1]$ is called a fuzzy number if $\tilde{a}$ is normal, convex fuzzy set, upper semi-continuous and $supp(a) = \overline{\{x \in \mathbb{R} : \tilde{a}(x) > 0\}}$ is compact, where $\overline{M}$ denotes the closure of M. In the rest of this paper we use E^1 to denote the fuzzy number space.

The $\alpha-$level set of $\tilde{a} \in E^1$ denoted by $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$, is defined by

$$\tilde{a}[\alpha] = \begin{cases} \{x \in \mathbb{R} : \tilde{a}(x) \geq \alpha\}, \text{ if } 0 < \alpha \leq 1, \\ \overline{\{x \in \mathbb{R} : \tilde{a}(x) > 0\}}, \text{ if } \ \alpha = 0. \end{cases}$$

Obviously, the α-level set $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$ is bounded closed interval in $\mathbb{R}$ for all $\alpha \in [0,1]$, where $a^l(\alpha)$ and $a^r(\alpha)$ denote the left-hand and right-hand endpoints of $\tilde{a}[\alpha]$, respectively. Any crisp number with value k can be regarded as a fuzzy number $\tilde{a}$ if its membership function is defined by,

$$\tilde{a}(x) = \begin{cases} 1 & , x = k, \\ 0 & , x \neq k. \end{cases}$$

Particularly,

$$\tilde{0}(x) = \begin{cases} 1 & , x = 0, \\ 0 & , x \neq 0. \end{cases}$$

For $\tilde{a}, \tilde{b} \in E^1, k \in R$, we can define the addition and scalar multiplication by using α-level set respectively, as

$$\begin{aligned} (\tilde{a} \oplus \tilde{b})[\alpha] &= \tilde{a}[\alpha] + \tilde{b}[\alpha] = \{x + y : x \in \tilde{a}[\alpha], y \in \tilde{b}[\alpha]\}, \\ (k \odot \tilde{a})[\alpha] &= k\tilde{a}[\alpha] = \{kx : x \in \tilde{a}[\alpha]\}. \end{aligned}$$

Where $\tilde{a}[\alpha] + \tilde{b}[\alpha]$ means the usual addition of two intervals of $\mathbb{R}$, and $k\tilde{a}[\alpha]$ means the usual product between a scalar and interval of $\mathbb{R}$. Furthermore, the opposite of the fuzzy number $\tilde{a}$ is $-\tilde{a}$, i.e., $-\tilde{a}(x) = \tilde{a}(-x)$, it means, $-\tilde{a}[\alpha] = [-a^r(\alpha), -a^l(\alpha)]$, in the case of $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$.

The binary operation "." in $\mathbb{R}$ can be extended to the binary operation "$\odot$" of two fuzzy numbers by using the extension principle. Let $\tilde{a}$ and $\tilde{b}$ be fuzzy numbers, then

$$(\tilde{a} \odot \tilde{b})(z) = \sup_{x \cdot y = z} \min\{\tilde{a}(x), \tilde{b}(y)\}.$$

Using α-level set, the product $(\tilde{a} \odot \tilde{b})$ is defined by

$$\begin{aligned} (\tilde{a} \odot \tilde{b})[\alpha] = \big[&\min\{a^l(\alpha)b^l(\alpha), a^l(\alpha)b^r(\alpha), a^r(\alpha)b^l(\alpha), a^r(\alpha)b^r(\alpha)\}, \\ &\max\{a^l(\alpha)b^l(\alpha), a^l(\alpha)b^r(\alpha), a^r(\alpha)b^l(\alpha), a^r(\alpha)b^r(\alpha)\}\big], \end{aligned}$$

in the case of, $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$ and $\tilde{b}[\alpha] = [b^l(\alpha), b^r(\alpha)]$.

The metric structure is given by the Hausdorff distance $D : E^1 \times E^1 \times \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$,

$$D(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} \max\{| a^l(\alpha) - b^l(\alpha) |, | a^r(\alpha) - b^r(\alpha) |\}.$$

We know that, D satisfies the following properties:

(1) (E^n, D) is a complete metric space,
(2) $D(\tilde{a} \oplus \tilde{c}, \tilde{b} \oplus \tilde{c}) = D(\tilde{a}, \tilde{b})$,
(3) $D(k \odot \tilde{a}, k \odot \tilde{b}) = |k| D(\tilde{a}, \tilde{b})$, where $\tilde{a}, \tilde{b}, \tilde{c} \in E^1$ and $k \in R$.

A special class of fuzzy numbers is the class of triangular fuzzy numbers. We say that the fuzzy number $\tilde{a}$ is triangular if $a^l(1) = a^r(1)$, $a^l(\alpha) = a^l(1) - (1 - \alpha)(a^l(1) - a^l(0))$ and $a^r(\alpha) = a^l(1) + (1 - \alpha)(a^r(0) - a^l(1))$. The triangular fuzzy number $\tilde{a}$ is generally denoted by $\tilde{a} = (a^l(0), a^l(1), a^r(0))$.

The following lemma gives the properties of the left and right-hand functions of its $\alpha-$level set of any fuzzy number $\tilde{a} \in E^1$.

Lemma 2.1. *(See [6]). Let $a^l : [0,1] \to \mathbb{R}$ and $a^r : [0,1] \to \mathbb{R}$ satisfy the conditions:*

C1: *a^l is a bounded increasing function,*
C2: *a^r is a bounded decreasing function,*
C3: *$a^l(1) \leq a^r(1)$,*

C4: $\lim_{\alpha \to k^-} a^l(\alpha) = a^l(k)$ *and* $\lim_{\alpha \to k^-} a^r(\alpha) = a^r(k)$, *for all* $0 < k \leq 1$,
C5: $\lim_{\alpha \to 0^+} a^l(\alpha) = a^l(0)$ *and* $\lim_{\alpha \to 0^+} a^r(\alpha) = a^r(0)$.

Then $\tilde{a} : \mathbb{R} \to [0,1]$ *defined by* $\tilde{a}(x) = \sup\{\alpha | a^l(\alpha) \leq x \leq a^r(\alpha)\}$ *is a fuzzy number with* $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$. *Moreover, if* $\tilde{a} : \mathbb{R} \to [0,1]$ *is a fuzzy number with* $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$, *then the functions* $a^l(\alpha)$ *and* $a^r(\alpha)$ *satisfy conditions* ***C1- C5***.

Definition 2.2. (H-difference [8]). Suppose that $\tilde{a}, \tilde{b} \in E^1$, where $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$ and $\tilde{b}[\alpha] = [b^l(\alpha), b^r(\alpha)]$ for all $\alpha \in [0,1]$, the H-difference of $\tilde{a}$ and $\tilde{b}$ is defined by

$$\tilde{a} \ominus_H \tilde{b} = \tilde{c} \iff \tilde{a} = \tilde{b} \oplus \tilde{c}.$$

Obviously, $\tilde{a} \ominus_H \tilde{a} = \tilde{0}$, and the α-level set of H-difference is

$$(\tilde{a} \ominus_H \tilde{b})[\alpha] = [a^l(\alpha) - b^l(\alpha), a^r(\alpha) - b^r(\alpha)], \quad \forall \alpha \in [0,1].$$

Definition 2.3. (Partial ordering [8]). Let $\tilde{a}, \tilde{b} \in E^1$, we write $\tilde{a} \preceq \tilde{b}$, if $a^l(\alpha) \leq b^l(\alpha)$ and $a^r(\alpha) \leq b^r(\alpha)$ for all $\alpha \in [0,1]$. We also write $\tilde{a} \prec \tilde{b}$, if $\tilde{a} \preceq \tilde{b}$ and there exists $\alpha_0 \in [0,1]$ such that $a^l(\alpha_0) < b^l(\alpha_0)$ or $a^r(\alpha_0) < b^r(\alpha_0)$. Furthermore, $\tilde{a} = \tilde{b}$, if $\tilde{a} \preceq \tilde{b}$ and $\tilde{a} \succeq \tilde{b}$. In other words, $\tilde{a} = \tilde{b}$, if $\tilde{a}[\alpha] = \tilde{b}[\alpha]$ for all $\alpha \in [0,1]$.

In the sequel, we say that $\tilde{a}, \tilde{b} \in E^1$ are *comparable* if either $\tilde{a} \preceq \tilde{b}$ or $\tilde{a} \succeq \tilde{b}$, and *non-comparable* otherwise.

From Now, $S \subseteq \mathbb{R}$ is considered to be a real subset.

Definition 2.4. (Fuzzy valued function [8]). The function $\tilde{f} : S \to E^1$ is called a fuzzy valued function if $\tilde{f}(x)$ is assign a fuzzy number for any $x \in S$. We also denote $\tilde{f}(x)[\alpha] = [f^l(x,\alpha), f^r(x,\alpha)]$, where $f^l(x,\alpha) = (\tilde{f}(x))^l(\alpha) = \min\{\tilde{f}(x)[\alpha]\}$ and $f^r(x,\alpha) = (\tilde{f}(x))^r(\alpha) = \max\{\tilde{f}(x)[\alpha]\}$. Therefore any fuzzy valued function $\tilde{f}$ may be understood by $f^l(x,\alpha)$ and $f^r(x,\alpha)$ being respectively a bounded increasing function of α and a bounded decreasing function of α for $\alpha \in [0,1]$. Also it holds $f^l(x,\alpha) \leq f^r(x,\alpha)$ for any $\alpha \in [0,1]$.

Definition 2.5. (Continuity of a fuzzy valued function [8]). We say that $\tilde{f} : S \to E^1$ is continuous at $x \in S$, if both $f^l(x,\alpha)$ and $f^r(x,\alpha)$ are continuous functions at $x \in S$ for all $\alpha \in [0,1]$.

Note that, $\tilde{C}[t_0, t_1]$ is used to denote the class of all fuzzy continuous functions on $[t_0, t_1]$.

Definition 2.6. (H-differentiability of a fuzzy valued function [7]). Let $s_0 \in S$ and h be such that $s_0 + h \in S$. A fuzzy valued function $\tilde{x} : S \to E^1$ is said to be H-differentiable at $s_0 \in S$ if and only if there exists a fuzzy number $\tilde{\dot{x}}(s_0)$ such that the limits (with respect to metric D)

$$\lim_{h \to 0^+} \frac{\tilde{x}(s_0 + h) \ominus_H \tilde{x}(s_0)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{\tilde{x}(s_0) \ominus_H \tilde{x}(s_0 - h)}{h},$$

both exist and are equal to $\tilde{\dot{x}}(s_0) \in E^1$. In this case $\tilde{\dot{x}}(s_0)$ is called the H-derivative of $\tilde{x}$ at s_0. If $\tilde{x}$ is H-differentiable at any $s \in S$, we call $\tilde{x}$ is H-differentiable over S.

Moreover, if a fuzzy valued function $\tilde{x} : S \to E^1$ is H-differentiable at $s_0 \in S$, then $\dot{x}^l(s,\alpha)$ and $\dot{x}^r(s,\alpha)$ are differentiable at $s_0 \in S$ for all $\alpha \in [0,1]$, and we have

$$\dot{\tilde{x}}(s_0)[\alpha] = \left[\dot{x}^l(s_0,\alpha), \dot{x}^r(s_0,\alpha)\right].$$

Definition 2.7. (See [6]). Let $\tilde{f} : [t_0,t_1] \to E^1$. We say that $\tilde{f}$ is Fuzzy-Riemann integrable to $I \in E^1$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[i,j];\xi\}$ of $[t_0,t_1]$ with the norms $\Delta(p) < \delta$, we have

$$D\left(\sum_P^{*}(i-j)\odot \tilde{f}(\xi); I\right) < \epsilon,$$

where $\sum^*$ denotes the fuzzy summation. We choose to write $I := (FR)\int_{t_0}^{t_1} \tilde{f}(t)dt$. Furthermore, for any $\alpha \in [0,1]$

$$(FR)\int_{t_0}^{t_1} \tilde{f}(t)[\alpha]dt = \left[(R)\int_{t_0}^{t_1} f^l(t,\alpha)dt, (R)\int_{t_0}^{t_1} f^r(t,\alpha)dt\right].$$

Definition 2.8. (Distance measure between fuzzy valued functions [8]). Suppose that $\tilde{f}, \tilde{g} : S \to E^1$ are two fuzzy functions. We define the distance measure between $\tilde{f}$ and $\tilde{g}$ by

$$\begin{aligned} D_{E^1}(\tilde{f}(x),\tilde{g}(x)) &= \sup_{\alpha\in[0,1]} \mathbf{H}(\tilde{f}(x)[\alpha],\tilde{g}(x)[\alpha]) \\ &= \max\{\sup_{z\in\tilde{f}(x)[\alpha]} d(z,\tilde{g}(x)[\alpha]), \sup_{y\in\tilde{g}(x)[\alpha]} d(\tilde{f}(x)[\alpha],y)\}, \ \ \forall x \in S, \end{aligned}$$

where $\mathbf{H}$ is the well-known Hausdorff metric on the family of all nonempty compact subsets of R, and

$$d(a,B) = \inf_{b\in B} d(a,b).$$

Moreover, we can define

$$\| \tilde{f}(x) \|_{E^1}^2 = D_{E^1}(\tilde{f}(x),\tilde{f}(x)), \ \ \forall x \in S,$$

for any $\tilde{f} : S \to E^1$.

3. Elements of Fuzzy Calculus of Variations

In this section, we recall some definitions and theorem of the calculus of variations in a fuzzy environment to be used in the subsequent discussion.

Definition 3.1. (Fuzzy increment [8]). Suppose that $\tilde{x}(.)$ and $\tilde{x}(.) \oplus \delta\tilde{x}(.)$ are fuzzy functions for which the fuzzy functional $\tilde{J}$ is defined. The increment of $\tilde{J}$, denoted by $\Delta\tilde{J}$, is

$$\Delta\tilde{J} := \tilde{J}(\tilde{x}\oplus\delta\tilde{x}) \ominus_H \tilde{J}(\tilde{x}), \tag{3.1}$$

where $\delta\tilde{x}(.)$ is known as the variation of $\tilde{x}(.)$.

Because the increment $\Delta\tilde{J}$ depends on the fuzzy functions $\tilde{x}$ and $\delta\tilde{x}$, we denote $\Delta\tilde{J}$ by $\Delta\tilde{J}(\tilde{x},\delta\tilde{x})$.

Definition 3.2. (Differentiability of a fuzzy functional [8]). Suppose that $\Delta\tilde{J}$ can be written as

$$\Delta\tilde{J}(\tilde{x},\delta\tilde{x}) := \delta\tilde{J}(\tilde{x},\delta\tilde{x}) \oplus \tilde{j}(\tilde{x},\delta\tilde{x}) \odot \parallel \delta\tilde{x} \parallel_{E^1}, \tag{3.2}$$

where $\delta\tilde{J}$ is linear in $\delta\tilde{x}$. We say that $\tilde{J}$ is differentiable with respect to $\tilde{x}$ if for any $\epsilon > 0$,

$$D_{E^1}(\tilde{j}(\tilde{x},\delta\tilde{x}),\tilde{0}) < \epsilon, \ \ as \parallel \delta\tilde{x}(.) \parallel_{E^1} \to \tilde{0}.$$

Definition 3.3. (Fuzzy relative minimum [8]). A fuzzy functional $\tilde{J}$ with domain $\tilde{C}[t_0,t_1]$, has a fuzzy relative minimizer $\tilde{x}^* = \tilde{x}^*(t)$, if the increment of $\tilde{J}$ is a fuzzy non-negative, that is,

$$\Delta\tilde{J} := \tilde{J}(\tilde{x}) \ominus_H \tilde{J}(\tilde{x}^*) \succeq \tilde{0}, \tag{3.3}$$

equivalent to

$$\tilde{J}(\tilde{x}) \succeq \tilde{J}(\tilde{x}^*), \tag{3.4}$$

for all fuzzy functions $\tilde{x} \in \tilde{C}[t_0,t_1]$.

It is clear that, from the definition of partial ordering, the inequality (3.4) holds iff

$$J^l(\tilde{x},\alpha) \geq J^l(\tilde{x}^*,\alpha), \ \ \text{and} \ \ J^r(\tilde{x},\alpha) \geq J^r(\tilde{x}^*,\alpha), \tag{3.5}$$

for all $\alpha \in [0,1]$ and all $\tilde{x},\tilde{x}^* \in \tilde{C}[t_0,t_1]$.

The following theorem is the fundamental theorem of the calculus of variations in fuzzy environment.

Theorem 3.4. [8] *Let $\tilde{x},\delta\tilde{x} \in \tilde{C}[t_0,t_1]$ be two fuzzy functions of $t \in [t_0,t_1]$, and $\tilde{J}(\tilde{x})$ is differentiable fuzzy functional of $\tilde{x}$. If $\tilde{x}^*$ is a fuzzy minimizer of $\tilde{J}$, then the variation of $\tilde{J}$ regardless of any boundary conditions must vanish on $\tilde{x}^*$, that is,*

$$\delta\tilde{J}(\tilde{x}^*,\delta\tilde{x}) = \tilde{0}, \tag{3.6}$$

for all admissible $\delta\tilde{x}$ having the property $\tilde{x} \oplus \delta\tilde{x} \in \tilde{C}[t_0,t_1]$.

It is obviously that the equality (3.6) holds if and only if

$$\delta J^l(\tilde{x}^*(t)[\alpha],\delta\tilde{x}(t)[\alpha],t,\alpha) = 0, \tag{3.7}$$
$$\delta J^r(\tilde{x}^*(t)[\alpha],\delta\tilde{x}(t)[\alpha],t,\alpha) = 0, \tag{3.8}$$

for all $\alpha \in [0,1], t \in [t_0,t_1]$ and all admissible $\delta\tilde{x}$ where

$$\delta\tilde{x}(t)[\alpha] = [\delta x^l(t,\alpha), \delta x^r(t,\alpha)].$$

4. Fuzzy Optimal Control Problem with State Conditions at The Final Time

In this section, first, we define the fuzzy optimal control problem with state conditions at the final time, then we derive the necessary conditions for this problem. Consider the fuzzy optimal control problem with state conditions at the final time is defined as

$$\begin{aligned}\min_{\tilde{u}} \tilde{J}(\tilde{u}) &= \tilde{\phi}(\tilde{x}(t_1), t_1) \oplus (FR)\int_{t_0}^{t_1} \tilde{f}(\tilde{x}(t), \tilde{u}(t), t)dt, \\ \text{subject to:} \quad \dot{\tilde{x}}(t) &= \tilde{g}(\tilde{x}(t), \tilde{u}(t), t), \\ \tilde{x}(t_0) &= \tilde{x}_0, \quad \tilde{x}(t_1) \text{ is free.}\end{aligned} \tag{4.1}$$

The term $\tilde{\phi}(\tilde{x}(t_1), t_1)$ is a goal with respect to the final position $\tilde{x}(t_1)$, and it is called a payoff term[1].

Where $\tilde{f}, \tilde{g} : E^1 \times E^1 \times \mathbb{R} \to E^1$ are fuzzy valued functions, the fuzzy state $\tilde{x}(t)$ and the fuzzy control $\tilde{u}(t)$ are functions of $t \in [t_0, t_1] \subseteq \mathbb{R}$, meanwhile, the fuzzy state function $\tilde{x}(t)$ is assumed to be H-differentiable with respect to $t \in [t_0, t_1]$. The integrand $\tilde{f}$ and the right hand side of the fuzzy differential equation $\tilde{g}$ are considered to have continuous first and second partial derivatives with respect to all of their arguments.

Definition 4.1. (Admissible fuzzy state[8]). We say that $\tilde{x}(t)$ is admissible, if it satisfies appropriate boundary condition and also is twice continuously differentiable with respect to $t \in [t_0, t_1]$.

In this paper, we consider an admissible fuzzy control $\tilde{u}(t)$ is not bounded[8].

4.1. **Derivation of Necessary Conditions-1.** In this part, we will derive the necessary conditions for problem (4.1) by using fuzzy variational approaches and the concept of parameterizing the fuzzy valued function and its differentiation and integration by the left and right-hand functions of its $\alpha-$level set by considering the following Remark.

Remark 4.2. In the following theorem, we assume that $J^l(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), \dot{\tilde{x}}(t), t, \alpha)$ (or $J^r(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), \dot{\tilde{x}}(t), t, \alpha)$) is stated in terms containing only $x^l(t,\alpha)$, $u^l(t,\alpha)$, $\lambda^l(t,\alpha)$ and $\dot{x}^l(t,\alpha)$ (or only $x^r(t,\alpha)$, $u^r(t,\alpha)$, $\lambda^r(t,\alpha)$ and $\dot{x}^r(t,\alpha)$). For short we just write $J^l_a(u^l, \alpha)$ or $J^r_a(u^r, \alpha)$.

Based on the restrictions imposed by the above Remark, we define type-1 fuzzy Hamiltonian function.

Definition 4.3. (Type-1 Fuzzy Hamiltonian Function). We define type-1 fuzzy Hamiltonian function as

$$\tilde{H}(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), t) = \tilde{f}(\tilde{x}(t), \tilde{u}(t), t) \oplus \tilde{\lambda}(t) \odot \tilde{g}(\tilde{x}(t), \tilde{u}(t), t), \tag{4.2}$$

thus

$$\tilde{H}(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), t)[\alpha] = \left[H^l(x^l, u^l, \lambda^l, t, \alpha), H^r(x^r, u^r, \lambda^r, t, \alpha)\right], \tag{4.3}$$

for any $\alpha \in [0,1]$ and where $H^l(x^l, u^l, \lambda^l, t, \alpha)$ and $H^r(x^r, u^r, \lambda^r, t, \alpha)$ are the classical Hamiltonian functions defined by

$$\begin{aligned}H^l(x^l, u^l, \lambda^l, t, \alpha) &= f^l(x^l, u^l, \lambda^l, t, \alpha) + \lambda^l g^l(x^l, u^l, \lambda^l, t, \alpha), \\ H^r(x^r, u^r, \lambda^r, t, \alpha) &= f^r(x^r, u^r, \lambda^r, t, \alpha) + \lambda^r g^r(x^r, u^r, \lambda^r, t, \alpha).\end{aligned}$$

The following theorem is considered to be the first part of our main results of this paper.

Theorem 4.4 (Necessary Conditions for problem (4.1))**.** *Assume that $\tilde{x}^*(t)$ be an admissible fuzzy state and $\tilde{u}^*(t)$ be an admissible fuzzy control. Then the necessary conditions for $\tilde{u}^*$ and $\tilde{x}^*$ to be an optimal solutions for (4.1) and for all $\alpha \in [0,1]$, $t \in [t_0, t_1]$ are:*

$$\dot{x}^{*^l}(t,\alpha) = \frac{\partial H^l}{\partial \lambda^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha), \tag{4.4}$$

$$\dot{x}^{*^r}(t,\alpha) = \frac{\partial H^r}{\partial \lambda^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha), \tag{4.5}$$

$$\dot{\lambda}^{*^l}(t,\alpha) = -\frac{\partial H^l}{\partial x^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha), \tag{4.6}$$

$$\dot{\lambda}^{*^r}(t,\alpha) = -\frac{\partial H^r}{\partial x^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha), \tag{4.7}$$

$$0 = \frac{\partial H^l}{\partial u^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha), \tag{4.8}$$

$$0 = \frac{\partial H^r}{\partial u^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha), \tag{4.9}$$

$$\lambda^l(t_1,\alpha) = \left.\frac{\partial \phi^l}{\partial x^l}\right|_{t=t_1}, \tag{4.10}$$

$$\lambda^r(t_1,\alpha) = \left.\frac{\partial \phi^r}{\partial x^r}\right|_{t=t_1}. \tag{4.11}$$

Proof. First we adopt fuzzy Lagrange multiplier function $\tilde{\lambda}(t)$ to form an augmented functional incorporating the constraints:

$$\begin{aligned} \tilde{J}_a(\tilde{u}) &= \tilde{\phi}(\tilde{x}(t_1), t_1) \oplus (FR)\int_{t_0}^{t_1} \Big[\tilde{f}(\tilde{x}(t), \tilde{u}(t), t) \oplus \tilde{\lambda}(t) \odot (\tilde{g}(\tilde{x}(t), \tilde{u}(t), t) \\ \ominus_H &\quad \dot{\tilde{x}}(t))\Big]\, dt. \end{aligned} \tag{4.12}$$

Without loss of generality, we consider

$$\begin{aligned} \tilde{J}_a(\tilde{u})[\alpha] &= \left[J_a^l(u^l,\alpha), J_a^r(u^r,\alpha)\right] = \Bigg[\phi^l(x^l(t_1), t_1) + (R)\int_{t_0}^{t_1} \left[f^l(x^l, u^l, t, \alpha)\right. \\ &+ \left.\lambda^l(t,\alpha)\left(g^l(x^l, u^l, t, \alpha) - \dot{x}^l(t,\alpha)\right)\right] dt, \phi^r(x^r(t_1), t_1) \\ &+ (R)\int_{t_0}^{t_1} \left[f^r(x^r, u^r, t, \alpha) + \lambda^r(t,\alpha)\left(g^r(x^r, u^r, t, \alpha) - \dot{x}^r(t,\alpha)\right)\right] dt\Bigg]. \end{aligned}$$

In the rest of the proof, we will ignore similar cases and only consider

$$J_a^l(u^l,\alpha) = \phi^l(x^l(t_1),t_1) + (R)\int_{t_0}^{t_1} \left[f^l(x^l,u^l,t,\alpha) + \lambda^l(t,\alpha)(g^l(x^l,u^l,t,\alpha) - \dot{x}^l(t,\alpha))\right] dt. \quad (4.13)$$

Using the definition of type-1 fuzzy Hamiltonian function (the left hand function), we rewrite Eqn.(4.13) as

$$J_a^l(u^l,\alpha) = \phi^l(x^l(t_1),t_1) + (R)\int_{t_0}^{t_1} \left[H^l(x^l,u^l,\lambda^l,t,\alpha) - \lambda^l(t,\alpha)\dot{x}^l(t,\alpha))\right] dt. \quad (4.14)$$

The variation of (4.14) is

$$\delta J_a^l(u^l,\alpha) = \frac{\partial \phi^l}{\partial x^l}\delta x^l\Big|_{t=t_1} + (R)\int_{t_0}^{t_1} \left(\frac{\partial H^l}{\partial x^l}\delta x^l + \frac{\partial H^l}{\partial u^l}\delta u^l + \frac{\partial H^l}{\partial \lambda^l}\delta\lambda^l - \dot{x}^l\delta\lambda^l - \lambda^l\delta\dot{x}^l\right) dt. \quad (4.15)$$

Integrating the last term on the RHS of (4.15) by parts, we have

$$\delta J_a^l(u^l,\alpha) = \left[\left(\frac{\partial \phi^l}{\partial x^l} - \lambda^l\right)\delta x^l\right]_{t=t_1} + \lambda^l\delta x^l\big|_{t=t_0} + (R)\int_{t_0}^{t_1} \left(\left(\frac{\partial H^l}{\partial x^l} + \dot{\lambda}^l\right)\delta x^l + \left(\frac{\partial H^l}{\partial \lambda^l} - \dot{x}^l\right)\delta\lambda^l + \frac{\partial H^l}{\partial u^l}\delta u^l\right) dt, \quad (4.16)$$

where δx^l, δu^l and $\delta\lambda^l$ are the variations of x^l, u^l and λ^l, respectively. Since $x^l(t_0,\alpha)$ is specified, then $\delta x^l(t_0,\alpha) = 0$, therefore the last term in (4.16) must be vanish.

Now, it is convenient to remove the terms in $\delta J_a^l(u^l,\alpha)$ involving δx^l and $\delta\lambda^l$, then we obtain

$$\begin{aligned}
\dot{x}^{*^l}(t,\alpha) &= \frac{\partial H^l}{\partial \lambda^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t,\alpha),\\
\dot{\lambda}^{*^l}(t,\alpha) &= -\frac{\partial H^l}{\partial x^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t,\alpha),\\
\lambda^l(t_1,\alpha) &= \frac{\partial \phi^l}{\partial x^l}\Big|_{t=t_1}.
\end{aligned}$$

It follows that

$$\delta J_a^l(u^l,\alpha) = (R)\int_{t_0}^{t_1} \left(\frac{\partial H^l}{\partial u^l}\delta u^l\right) dt.$$

u^{*^l} is an extremal if $\delta J_a^l(u^{*^l},\alpha) = 0$, for all $\alpha \in [0,1]$, thus, the necessary condition for u^{*^l} to be an extremal is

$$\frac{\partial H^l}{\partial u^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t,\alpha) = 0.$$

By the same manner, u^{*^r} is an extremal if the variation of J_a^r is zero, i.e., $\delta J_a^r(u^{*^r}, \alpha) = 0$. Then, for all $\alpha \in [0,1]$ and $t \in [0,1]$, we will obtain

$$\begin{aligned}
\dot{x}^{*^r}(t,\alpha) &= \frac{\partial H^r}{\partial \lambda^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha), && (4.17)\\
\dot{\lambda}^{*^r}(t,\alpha) &= -\frac{\partial H^r}{\partial x^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha), && (4.18)\\
0 &= \frac{\partial H^r}{\partial u^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha), && (4.19)\\
\lambda^r(t_1,\alpha) &= \left.\frac{\partial \phi^r}{\partial x^r}\right|_{t=t_1}. && (4.20)
\end{aligned}$$

The above equations form a set of necessary conditions that the left and right-hand functions of its $\alpha-$level set of optimal fuzzy control $\tilde{u}^*$ and corresponding optimal fuzzy state $\tilde{x}^*$ must satisfy. □

Note that if a fuzzy state variable $\tilde{x}(t)$ satisfies $\tilde{x}(t_0) = \tilde{x}_0$ and $\tilde{x}(t_1) = \tilde{x}_1$ both fixed, then $\tilde{\lambda}(t)$ has no boundary conditions [8].

4.2. **Derivation of Necessary Conditions-2.** In this part, we will consider the following optimal control problem

$$\begin{aligned}
\min_{\tilde{u}} \tilde{J}(\tilde{x}, \tilde{u}) &= (FR)\int_{t_0}^{t_1} \tilde{f}(\tilde{x}(t), \tilde{u}(t), t)dt,\\
\text{subject to:} \quad \dot{\tilde{x}}(t) &= \tilde{g}(\tilde{x}(t), \tilde{u}(t), t), && (4.21)\\
\tilde{x}(t_0) &= \tilde{x}_0, \quad \tilde{x}(t_1) \text{ is free.}
\end{aligned}$$

Here we consider $\tilde{\phi}(\tilde{x}(t_1), t_1) = \tilde{0}$ in order to simplify the result presentations.

We say that an admissible fuzzy curve $(\tilde{x}^*, \tilde{u}^*)$ is solution of problem (4.21), if for all admissible curve $(\tilde{x}, \tilde{u})$ of problem (4.21) satisfy the inequality (3.4) in the definition (3.3)

$$\tilde{J}(\tilde{x}^*, \tilde{u}^*) \preceq \tilde{J}(\tilde{x}, \tilde{u}).$$

We know that, from the definition of partial ordering, the above inequality holds if and only if

$$J^l(x^{*^l}, x^{*^r}, u^{*^l}, u^{*^r}, t, \alpha) \leq J^l(x^l, x^r, u^l, u^r, t, \alpha), \tag{4.22}$$

and

$$J^r(x^{*^l}, x^{*^r}, u^{*^l}, u^{*^r}, t, \alpha) \leq J^r(x^l, x^r, u^l, u^r, t, \alpha), \tag{4.23}$$

for all $\alpha \in [0,1]$, where the $\alpha-$level set of fuzzy curves $\tilde{x}^*$, $\tilde{x}$, $\tilde{u}^*$ and $\tilde{u}$ are

$$\begin{aligned}
\tilde{x}^*(t)[\alpha] &= \left[x^{*^l}(t,\alpha), x^{*^r}(t,\alpha)\right], \quad \tilde{x}(t)[\alpha] = \left[x^l(t,\alpha), x^r(t,\alpha)\right],\\
\tilde{u}^*(t)[\alpha] &= \left[u^{*^l}(t,\alpha), u^{*^r}(t,\alpha)\right], \quad \tilde{u}(t)[\alpha] = \left[u^l(t,\alpha), u^r(t,\alpha)\right],
\end{aligned}$$

respectively.

Without the restrictions imposed by Remark 4.2, we introduce the new derivation of necessary conditions for the problem (4.21).

Remark 4.5. In the following theorem, we consider $J^l(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), \tilde{\dot{x}}(t), t, \alpha)$(or $J^r(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), \tilde{\dot{x}}(t), t, \alpha)$) is stated in terms containing $x^l(t,\alpha)$, $x^r(t,\alpha)$, $u^l(t,\alpha)$, $u^r(t,\alpha)$, $\lambda^l(t,\alpha)$, $\lambda^r(t,\alpha)$, $\dot{x}^l(t,\alpha)$ and $\dot{x}^r(t,\alpha)$.

According to Remark 4.5, we define type-2 fuzzy Hamiltonian function.

Definition 4.6. (Type-2 Fuzzy Hamiltonian Function). We define type-2 fuzzy Hamiltonian function as

$$\begin{aligned} \mathbb{H}(x^l, x^r, u^l, u^r, \lambda^l, \lambda^r, t, \alpha) &= f^l(x^l, x^r, u^l, u^r, t, \alpha) + f^r(x^l, x^r, u^l, u^r, t, \alpha) \\ &+ \lambda^l g^l(x^l, x^r, u^l, u^r, t, \alpha) + \lambda^r g^r(x^l, x^r, u^l, u^r, t, \alpha). \end{aligned}$$

Now we are in the position to state the second part of the main results of this paper in the following theorem.

Theorem 4.7 (Necessary Conditions for problem (4.21))**.** *Assume that $\tilde{x}^*(t)$ be an admissible fuzzy state and $\tilde{u}^*(t)$ be an admissible fuzzy control. Then the necessary conditions for $\tilde{u}^*$ and $\tilde{x}^*$ to be an optimal solutions for (4.21) and for all $\alpha \in [0,1]$, $t \in [t_0, t_1]$ are:*

$$\dot{\lambda}^{*^l}(t,\alpha) = -\frac{\partial \mathbb{H}}{\partial x^l}(x^{*^l}, x^{*^r}, u^{*^l}, u^{*^r}, \lambda^{*^l}, \lambda^{*^r}, t, \alpha), \tag{4.24}$$

$$\dot{\lambda}^{*^r}(t,\alpha) = -\frac{\partial \mathbb{H}}{\partial x^r}(x^{*^l}, x^{*^r}, u^{*^l}, u^{*^r}, \lambda^{*^l}, \lambda^{*^r}, t, \alpha), \tag{4.25}$$

$$0 = \frac{\partial \mathbb{H}}{\partial u^l}(x^{*^l}, x^{*^r}, u^{*^l}, u^{*^r}, \lambda^{*^l}, \lambda^{*^r}, t, \alpha), \tag{4.26}$$

$$0 = \frac{\partial \mathbb{H}}{\partial u^r}(x^{*^l}, x^{*^r}, u^{*^l}, u^{*^r}, \lambda^{*^l}, \lambda^{*^r}, t, \alpha), \tag{4.27}$$

$$0 = \lambda^{*^l}(t_1, \alpha), \tag{4.28}$$

$$0 = \lambda^{*^r}(t_1, \alpha). \tag{4.29}$$

Proof. Let us first consider the variation of u^l, u^r and the variation of it's corresponding states x^l, x^r as

$$u^l = u^{*^l} + \delta u^l, \quad u^r = u^{*^r} + \delta u^r, \quad x^l = x^{*^l} + \delta x^l, \quad x^r = x^{*^r} + \delta x^r,$$

respectively. The increment of $\tilde{J}$ is

$$\Delta \tilde{J} = (FR)\int_{t_0}^{t_1} \tilde{f}(\tilde{x}^* \oplus \delta\tilde{x}, \tilde{u}^* \oplus \delta\tilde{u}, t)dt \ominus_H (FR)\int_{t_0}^{t_1} \tilde{f}(\tilde{x}^*, \tilde{u}^*, t)dt.$$

Let $\Delta\tilde{J}[\alpha]=\left[\Delta J^l,\Delta J^r\right]$. Using H-difference, then one gets

$$\begin{aligned}
\Delta J^l &= (R)\int_{t_0}^{t_1} f^l\left[\tilde{x}^*\oplus\delta\tilde{x},\tilde{u}^*\oplus\delta\tilde{u}\right][\alpha]dt-(R)\int_{t_0}^{t_1} f^l\left[\tilde{x}^*,\tilde{u}^*\right][\alpha]dt,\\
&= (R)\int_{t_0}^{t_1} f^l(x^{*^l}+\delta x^l,x^{*^r}+\delta x^r,u^{*^l}+\delta u^l,u^{*^r}+\delta u^r,t,\alpha)dt\\
&- (R)\int_{t_0}^{t_1} f^l(x^{*^l},x^{*^r},u^{*^l},u^{*^r},t,\alpha)dt,\\
\Delta J^r &= (R)\int_{t_0}^{t_1} f^r\left[\tilde{x}^*\oplus\delta\tilde{x},\tilde{u}^*\oplus\delta\tilde{u}\right][\alpha]dt-(R)\int_{t_0}^{t_1} f^r\left[\tilde{x}^*,\tilde{u}^*\right][\alpha]dt,\\
&= (R)\int_{t_0}^{t_1} f^r(x^{*^l}+\delta x^l,x^{*^r}+\delta x^r,u^{*^l}+\delta u^l,u^{*^r}+\delta u^r,t,\alpha)dt\\
&- (R)\int_{t_0}^{t_1} f^r(x^{*^l},x^{*^r},u^{*^l},u^{*^r},t,\alpha)dt,
\end{aligned}$$

where

$$\begin{aligned}
\left[\tilde{x}^*\oplus\delta\tilde{x},\tilde{u}^*\oplus\delta\tilde{u}\right][\alpha] &= (x^{*^l}+\delta x^l,x^{*^r}+\delta x^r,u^{*^l}+\delta u^l,u^{*^r}+\delta u^r,t,\alpha),\\
\left[\tilde{x}^*,\tilde{u}^*\right][\alpha] &= (x^{*^l},x^{*^r},u^{*^l},u^{*^r},t,\alpha).
\end{aligned}$$

We know that, $\tilde{J}(\tilde{x}^*,\tilde{u}^*)\preceq\tilde{J}(\tilde{x},\tilde{u})$ if and only if the inequalities (4.22) and (4.23) are hold, it means that $(\left[\tilde{x}^*,\tilde{u}^*\right][\alpha])$ is an optimal solution for the crisp functions J^l and J^r.

From the classical theory of optimal control, if u^{*^l} and u^{*^r} are optimal, then it is necessary that the first variation of J^l and J^r are zero. In order to find the first variation of J^l and J^r, we need to evaluate the derivatives in the integrand of J^l and J^r along the optimal trajectory, then we obtain

$$\begin{aligned}
\Delta J^l &= (R)\int_{t_0}^{t_1}\left[\frac{\partial f^l}{\partial x^l}\delta x^l+\frac{\partial f^l}{\partial x^r}\delta x^r+\frac{\partial f^l}{\partial u^l}\delta u^l+\frac{\partial f^l}{\partial u^r}\delta u^r\right]dt+O((\delta u^{*^l})^2)\\
&+ O((\delta u^{*^r})^2),\\
\Delta J^r &= (R)\int_{t_0}^{t_1}\left[\frac{\partial f^r}{\partial x^l}\delta x^l+\frac{\partial f^r}{\partial x^r}\delta x^r+\frac{\partial f^r}{\partial u^l}\delta u^l+\frac{\partial f^r}{\partial u^r}\delta u^r\right]dt+O((\delta u^{*^l})^2)\\
&+ O((\delta u^{*^r})^2).
\end{aligned}$$

Subsequently, on optimal trajectories, the first variation of J^l and J^r are

$$\begin{aligned}
\delta J^l &= (R)\int_{t_0}^{t_1}\left[\frac{\partial f^l}{\partial x^l}\delta x^l+\frac{\partial f^l}{\partial x^r}\delta x^r+\frac{\partial f^l}{\partial u^l}\delta u^l+\frac{\partial f^l}{\partial u^r}\delta u^r\right]dt=0,\\
\delta J^r &= (R)\int_{t_0}^{t_1}\left[\frac{\partial f^r}{\partial x^l}\delta x^l+\frac{\partial f^r}{\partial x^r}\delta x^r+\frac{\partial f^r}{\partial u^l}\delta u^l+\frac{\partial f^r}{\partial u^r}\delta u^r\right]dt=0,
\end{aligned}$$

for all variations. Now we are ready to introduce the fuzzy Lagrange multiplier function $\tilde{\lambda}(t)$ by considering the integral

$$\tilde{\psi} = (FR)\int_{t_0}^{t_1} \tilde{\lambda}(t) \odot \left(\tilde{g}(\tilde{x}, \tilde{u}, t) \ominus_H \dot{\tilde{x}}(t)\right) dt.$$

Without loss of generality, we consider the $\alpha-$level set of $\tilde{\psi}$ are

$$\psi^l = (R)\int_{t_0}^{t_1} -\lambda^l(\dot{x}^l - g^l)dt, \tag{4.30}$$

and

$$\psi^r = (R)\int_{t_0}^{t_1} -\lambda^r(\dot{x}^r - g^r)dt, \tag{4.31}$$

respectively. In the remaining of the proof we will ignore the similar arguments. We start by computing the variation of (4.30)

$$\begin{aligned}
\delta\psi^l &= (R)\int_{t_0}^{t_1} -\delta\lambda^l(\dot{x}^l - g^l) - \lambda^l\left[\delta\dot{x}^l - \left(\frac{\partial g^l}{\partial x^l}\delta x^l + \frac{\partial g^l}{\partial x^r}\delta x^r + \frac{\partial g^l}{\partial u^l}\delta u^l\right.\right. \\
&+ \left.\left.\frac{\partial g^l}{\partial u^r}\delta u^r\right)\right] dt, \\
&= (R)\int_{t_0}^{t_1} \left[\lambda^l\left(\frac{\partial g^l}{\partial x^l}\delta x^l + \frac{\partial g^l}{\partial x^r}\delta x^r + \frac{\partial g^l}{\partial u^l}\delta u^l + \frac{\partial g^l}{\partial u^r}\delta u^r\right)\right. \\
&- \left.\lambda^l\delta\dot{x}^l\right] dt.
\end{aligned} \tag{4.32}$$

Integrating the last term on the RHS of (4.32) by parts and because $\tilde{x}(t_0)$ is specified i.e. $\delta x^l(t_0) = 0$, then one gets

$$\begin{aligned}
\delta\psi^l &= (R)\int_{t_0}^{t_1} \left(\left(\dot{\lambda}^l + \lambda^l\frac{\partial g^l}{\partial x^l}\right)\delta x^l + \lambda^l\left(\frac{\partial g^l}{\partial x^r}\delta x^r + \frac{\partial g^l}{\partial u^l}\delta u^l + \frac{\partial g^l}{\partial u^r}\delta u^r\right)\right) dt \\
&- \lambda^l(t_1)\delta x^l(t_1).
\end{aligned}$$

Similarly, when we consider (4.31) with $\delta x^r(t_0) = 0$, we arrive at

$$\begin{aligned}
\delta\psi^r &= (R)\int_{t_0}^{t_1} \left(\left(\dot{\lambda}^r + \lambda^r\frac{\partial g^r}{\partial x^r}\right)\delta x^r + \lambda^r\left(\frac{\partial g^r}{\partial x^l}\delta x^l + \frac{\partial g^r}{\partial u^l}\delta u^l + \frac{\partial g^r}{\partial u^r}\delta u^r\right)\right) dt \\
&- \lambda^r(t_1)\delta x^r(t_1).
\end{aligned}$$

Since $\psi^l = 0$ and $\psi^r = 0$ for all u^l and u^r then $\delta\psi^l = 0$ and $\delta\psi^r = 0$. Further, we can replace the conditions $\delta J^l = 0$ and $\delta J^r = 0$ by $\delta J^l + \delta\psi^l = 0$ and $\delta J^r + \delta\psi^r = 0$, respectively, as in [9]. Then we have

$$\begin{aligned}
\delta J^l + \delta\psi^l &= (R)\int_{t_0}^{t_1} \left(\frac{\partial f^l}{\partial x^l} + \lambda^l\frac{\partial g^l}{\partial x^l} + \dot{\lambda}^l\right)\delta x^l + \left(\frac{\partial f^l}{\partial x^r} + \lambda^l\frac{\partial g^l}{\partial x^r}\right)\delta x^r + \left(\frac{\partial f^l}{\partial u^l}\right. \\
&+ \left.\lambda^l\frac{\partial g^l}{\partial u^l}\right)\delta u^l + \left(\frac{\partial f^l}{\partial u^r} + \lambda^l\frac{\partial g^l}{\partial u^r}\right)\delta u^r dt - \lambda^l(t_1)\delta x^l(t_1) = 0.
\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}\delta J^r + \delta \psi^r &= (R)\int_{t_0}^{t_1} \left(\frac{\partial f^r}{\partial x^l} + \lambda^r \frac{\partial g^r}{\partial x^l}\right)\delta x^l + \left(\frac{\partial f^r}{\partial x^r} + \lambda^r \frac{\partial g^r}{\partial x^r} + \dot{\lambda}^r\right)\delta x^r + \left(\frac{\partial f^r}{\partial u^l}\right. \\ &+ \left.\lambda^r \frac{\partial g^r}{\partial u^l}\right)\delta u^l + \left(\frac{\partial f^r}{\partial u^r} + \lambda^r \frac{\partial g^r}{\partial u^r}\right)\delta u^r dt - \lambda^r(t_1)\delta x^r(t_1) = 0. \quad (4.34)\end{aligned}$$

Summing Eqs. (4.33) and (4.34), and then using the definition of type-2 fuzzy Hamiltonian function, we have

$$\begin{aligned}0 &= (R)\int_{t_0}^{t_1} \left(\frac{\partial \mathbb{H}}{\partial x^l} + \dot{\lambda}^l\right)\delta x^l + \left(\frac{\partial \mathbb{H}}{\partial x^r} + \dot{\lambda}^r\right)\delta x^r + \frac{\partial \mathbb{H}}{\partial u^l}\delta u^l + \frac{\partial \mathbb{H}}{\partial u^r}\delta u^r dt \\ &- \left(\lambda^l(t_1)\delta x^l(t_1) + \lambda^r(t_1)\delta x^r(t_1)\right). \quad (4.35)\end{aligned}$$

Since $\tilde{x}(t_1)$ is not specified, accordingly, we choose $\lambda^l(t_1) = 0$ and $\lambda^r(t_1) = 0$, therefore the necessary conditions follow. □

Now we are in the position to set up the last part of our main results of this paper.

4.3. **The Strong and Weak Solutions.** As we shown above, Theorem 4.4 and Theorem 4.7 provided the necessary conditions for $\tilde{u}^*$ and $\tilde{x}^*$ to be an optimal solutions of problem (4.1) and problem (4.21), respectively. We know that Lemma(2.1) gives the properties of the left and right-hand functions of its $\alpha-$level set of any fuzzy number. Directly, the left and right-hand functions of the $\alpha-$level set of optimal fuzzy control $\tilde{u}^*$ and optimal fuzzy state $\tilde{x}^*$ must fulfill those properties. Therefore, to assure that the solutions of problem (4.1) and problem (4.21) are fuzzy functions, we propose the concept of strong and weak solutions of our problems. Furthermore, $\tilde{u}^*(t)$ and $\tilde{x}^*(t)$ are fuzzy numbers with $\tilde{u}^*(t)[\alpha] = \left[u^{*^l}(t,\alpha), u^{*^r}(t,\alpha)\right]$ and $\tilde{x}^*(t)[\alpha] = \left[x^{*^l}(t,\alpha), x^{*^r}(t,\alpha)\right]$, if $u^{*^l}(t,\alpha), u^{*^r}(t,\alpha), x^{*^l}(t,\alpha)$ and $x^{*^r}(t,\alpha)$ satisfy their related properties defined in **C1-C5** of Lemma 2.1. In the following definition, based on the conditions **C1** and **C2** of Lemma 2.1, we introduce the definition of strong and weak solutions of the fuzzy optimal control problem with state conditions at the final time defined in (4.1) and (4.21), respectively.

Definition 4.8. (Strong and Weak Solutions).

(1) (Strong Solution). We say that $\tilde{u}^*(t)[\alpha]$ and $\tilde{x}^*(t)[\alpha]$ are strong solutions of problem (4.1) (or problem (4.21)) if $u^{*^l}(t,\alpha), u^{*^r}(t,\alpha), x^{*^l}(t,\alpha)$ and $x^{*^r}(t,\alpha)$ obtained from (4.4)-(4.11) (or from (4.24)-(4.29)) satisfy its related properties defined in the conditions **C1** and **C2** of Lemma 2.1, for all $t \in [t_0, t_1]$ and $\alpha \in [0,1]$.

(2) (Weak Solution). We say that $\tilde{u}^*(t)[\alpha]$ and $\tilde{x}^*(t)[\alpha]$ are weak solutions of problem (4.1) (or problem (4.21)) if $u^{*^l}(t,\alpha), u^{*^r}(t,\alpha), x^{*^l}(t,\alpha)$ and $x^{*^r}(t,\alpha)$ obtained from (4.4)-(4.11) (or from (4.24)-(4.29)) do not satisfy its related properties in the conditions **C1** and **C2** of Lemma 2.1,

meanwhile, we define $\tilde{u}^*(t)[\alpha]$ and $\tilde{x}^*(t)[\alpha]$, respectively, as

$$\tilde{u}^*(t)[\alpha] = \begin{cases} [2u^{*^r}(t,1) - u^{*^l}(t,\alpha), u^{*^r}(t,\alpha)], u^{*^l}, u^{*^r} \text{are decreasing functions of } \alpha, \\ [u^{*^l}(t,\alpha), 2u^{*^l}(t,1) - u^{*^r}(t,\alpha)], u^{*^l}, u^{*^r} \text{are increasing functions of } \alpha, \\ [u^{*^r}(t,\alpha), u^{*^l}(t,\alpha)], u^{*^l} \text{is decreasing and } u^{*^r} \text{is an increasing of } \alpha \end{cases}$$

and

$$\tilde{x}^*(t)[\alpha] = \begin{cases} [2x^{*^r}(t,1) - x^{*^l}(t,\alpha), x^{*^r}(t,\alpha)], x^{*^l}, x^{*^r} \text{are decreasing functions of } \alpha, \\ [x^{*^l}(t,\alpha), 2x^{*^l}(t,1) - x^{*^r}(t,\alpha)], x^{*^l}, x^{*^r} \text{are increasing functions of } \alpha, \\ [x^{*^r}(t,\alpha), x^{*^l}(t,\alpha)], x^{*^l} \text{is decreasing and } x^{*^r} \text{is an increasing of } \alpha. \end{cases}$$

for all $t \in [t_0, t_1]$ and $\alpha \in [0, 1]$.

In the next section, we will give three examples that can serve to illustrate our main results.

5. Illustrations

Example 5.1. Find the fuzzy control that minimize

$$\begin{aligned} \tilde{J}(\tilde{u}(t)) &= (FR)\int_0^1 \left(\tilde{x}(t) \ominus_H \tilde{u}^2(t)\right) dt, \\ \text{subject to:} \quad \dot{\tilde{x}}(t) &= \tilde{u}(t) \oplus (0,1,3)\tilde{x}(t), \quad t \in [0,1], \qquad (5.1) \\ \tilde{x}(0) &= \tilde{1} = (1,1,1), \quad \tilde{x}(t_1) \text{ is free.} \end{aligned}$$

Solution. First we formulate the type-1 and type-2 fuzzy Hamiltonian functions as:

$$\tilde{H}(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), t) = (\tilde{x}(t) \ominus_H \tilde{u}^2(t)) \oplus \tilde{\lambda}(t) \odot [\tilde{u}(t) \oplus (0,1,3)\tilde{x}(t)], \qquad (5.2)$$

$$\begin{aligned} \mathbb{H}(x^l, x^r, u^l, u^r, \lambda^l, \lambda^r, t, \alpha) &= \left(x^l - u^{l^2}\right) + \left(x^r - u^{r^2}\right) + \lambda^l\left(u^l + \alpha x^l\right) \\ &+ \left(u^r + (3 - 2\alpha)x^r\right), \qquad (5.3) \end{aligned}$$

respectively. Then we have

$$\begin{aligned} H^l(x^l, u^l, \lambda^l, t, \alpha) &= (x^l(t,\alpha) - u^{l^2}(t,\alpha)) + \lambda^l(t,\alpha)[u^l(t,\alpha) + \alpha x^l(t,\alpha)], \qquad (5.4) \\ H^r(x^r, u^r, \lambda^r, t, \alpha) &= (x^r(t,\alpha) - u^{r^2}(t,\alpha)) + \lambda^r(t,\alpha)[u^r(t,\alpha) \\ &+ (3 - 2\alpha)x^r(t,\alpha)], \qquad (5.5) \end{aligned}$$

and

$$\begin{aligned} \left[\phi^l(x^l(1), 1), \phi^r(x^r(1), 1)\right] &= [0, 0], \\ \left[\left.\frac{\partial \phi^l}{\partial x^l}\right|_{t=1}, \left.\frac{\partial \phi^r}{\partial x^r}\right|_{t=1}\right] &= [0, 0]. \qquad (5.6) \end{aligned}$$

Theorem(4.4) and Theorem(4.7) give the necessary conditions for optimality as

$$\dot{x}^l(t,\alpha) = \frac{\partial H^l}{\partial \lambda^l} = u^l(t,\alpha) + \alpha x^l(t,\alpha), \tag{5.7}$$

$$\dot{\lambda}^l(t,\alpha) = -\frac{\partial H^l}{\partial x^l} = -(1 + \alpha\lambda^l(t,\alpha)), \tag{5.8}$$

$$0 = \frac{\partial H^l}{\partial u^l} = -2u^l(t,\alpha) + \lambda^l(t,\alpha), \tag{5.9}$$

$$\lambda^l(1,\alpha) = \left.\frac{\partial \phi^l}{\partial x^l}\right|_{t=1} \Rightarrow \lambda^l(1,\alpha) = 0. \tag{5.10}$$

We start by solving the DE (5.8) with the condition (5.10), then we obtain

$$\lambda^{*^l}(t,\alpha) = \frac{e^{\alpha(1-t)} - 1}{\alpha}. \tag{5.11}$$

Substituting (5.11) into (5.9) gives

$$u^{*^l}(t,\alpha) = \frac{e^{\alpha(1-t)} - 1}{2\alpha}. \tag{5.12}$$

Substituting (5.12) into differential equation (5.7), and then solve it with the initial condition $x^{*^l}(0,\alpha) = 1$, then we obtain

$$x^{*^l}(t,\alpha) = \frac{(4\alpha^2 - 2)e^{\alpha t} + e^{\alpha(t+1)} - e^{\alpha(1-t)} + 2}{4\alpha^2}. \tag{5.13}$$

By using the same arguments and considering (5.19), we will obtain

$$u^{*^r}(t,\alpha) = \frac{e^{(3-2\alpha)(1-t)} - 1}{2(3-2\alpha)}, \tag{5.14}$$

$$x^{*^r}(t,\alpha) = \frac{(4(3-2\alpha)^2 - 2)e^{(3-2\alpha)t} + e^{(3-2\alpha)(t+1)} - e^{(3-2\alpha)(1-t)} + 2}{4(3-2\alpha)^2}. \tag{5.15}$$

As we see in Figure(1) $u^{*^l}(t,\alpha)$ and $x^{*^l}(t,\alpha)$ are an increasing functions of α, also $u^{*^r}(t,\alpha)$ and $x^{*^r}(t,\alpha)$ are decreasing functions of α. And for all $t \in [0,1]$,

$$u^{*^l}(t,1) = u^{*^r}(t,1) = -\frac{1}{2} + \frac{1}{2}e^{(1-t)},$$

$$x^{*^l}(t,1) = x^{*^r}(t,1) = \frac{1}{2}e^t + \frac{1}{4}e^{(t+1)} - \frac{1}{4}e^{(1-t)} + \frac{1}{2}.$$

Therefore, $u^{*^l}(t,\alpha), u^{*^r}(t,\alpha), x^{*^l}(t,\alpha)$ and $x^{*^r}(t,\alpha)$ are satisfies the conditions of Lemma(2.1). Now the α-level set of optimal fuzzy control $\tilde{u}^*(t)$ and optimal fuzzy state $\tilde{x}^*(t)$ are characterized, respectively, by

$$\begin{aligned}
\tilde{u}^*(t)[\alpha] &= \left[u^{*^l}(t,\alpha), u^{*^r}(t,\alpha)\right] = \left[\frac{e^{\alpha(1-t)}-1}{2\alpha}, \frac{e^{(3-2\alpha)(1-t)}-1}{2(3-2\alpha)}\right],\\
\tilde{x}^*(t)[\alpha] &= \left[x^{*^l}(t,\alpha), x^{*^r}(t,\alpha)\right],\\
&= \left[\frac{(4\alpha^2-2)e^{\alpha t}+e^{\alpha(t+1)}-e^{\alpha(1-t)}+2}{4\alpha^2},\right.\\
&\quad \left.\frac{(4(3-2\alpha)^2-2)e^{(3-2\alpha)t}+e^{(3-2\alpha)(t+1)}-e^{(3-2\alpha)(1-t)}+2}{4(3-2\alpha)^2}\right].
\end{aligned}$$

Therefore, the above solutions are strong solutions of problem (5.1).

Example 5.2. Find the fuzzy control that minimize

$$\begin{aligned}
\tilde{J}(\tilde{u}(t)) &= \tilde{x}(1) \oplus (FR)\int_0^1 \tilde{u}^2(t)dt,\\
\text{subject to:}\quad \dot{\tilde{x}}(t) &= \tilde{u}(t) \ominus_H (0,1,3)\tilde{x}(t), \quad t\in[0,1], \qquad (5.16)\\
\tilde{x}(0) &= \tilde{1} = (1,1,1), \quad \tilde{x}(t_1) \text{ is free.}
\end{aligned}$$

Solution. First we formulate type-1 fuzzy Hamiltonian function as

$$\tilde{H}(\tilde{x}(t),\tilde{u}(t),\tilde{\lambda}(t),t) = \tilde{u}^2(t) \oplus \tilde{\lambda}(t) \odot [\tilde{u}(t) \ominus_H (0,1,3)\tilde{x}(t)], \tag{5.17}$$

then we have

$$H^l(x^l,u^l,\lambda^l,t,\alpha) = u^{l^2}(t,\alpha) + \lambda^l(t,\alpha)\left[u^l(t,\alpha) - (3-2\alpha)x^l(t,\alpha)\right], \tag{5.18}$$

$$H^r(x^r,u^r,\lambda^r,t,\alpha) = u^{r^2}(t,\alpha) + \lambda^r(t,\alpha)\left[u^r(t,\alpha) - \alpha x^r(t,\alpha)\right], \tag{5.19}$$

and

$$\begin{aligned}
\left[\phi^l(x^l(1),1), \phi^r(x^r(1),1)\right] &= [x^l(1), x^r(1)],\\
\left[\left.\frac{\partial\phi^l}{\partial x^l}\right|_{t=1}, \left.\frac{\partial\phi^r}{\partial x^r}\right|_{t=1}\right] &= [1,1]. \qquad (5.20)
\end{aligned}$$

Theorem 4.4 give the necessary conditions for optimality as

$$\dot{x}^l(t,\alpha) = \frac{\partial H^l}{\partial\lambda^l} = u^l(t,\alpha) - (3-2\alpha)x^l(t,\alpha), \tag{5.21}$$

$$\dot{\lambda}^l(t,\alpha) = -\frac{\partial H^l}{\partial x^l} = (3-2\alpha)\lambda^l(t,\alpha), \tag{5.22}$$

$$0 = \frac{\partial H^l}{\partial u^l} = 2u^l(t,\alpha) + \lambda^l(t,\alpha), \tag{5.23}$$

$$\lambda^l(1,\alpha) = \left.\frac{\partial\phi^l}{\partial x^l}\right|_{t=1} \Rightarrow \lambda^l(1,\alpha) = 1. \tag{5.24}$$

We start by solving the DE (5.22) with the condition (5.24), then we obtain

$$\lambda^{*^l}(t,\alpha) = e^{(3-2\alpha)(t-1)}. \tag{5.25}$$

Substituting (5.25) into (5.23) gives

$$u^{*^l}(t,\alpha) = -\frac{1}{2}e^{(3-2\alpha)(t-1)}. \tag{5.26}$$

Substituting (5.26) into differential equation (5.21), and then solve it with the initial condition $x^{*^l}(0,\alpha) = 1$, then we obtain

$$x^{*^l}(t,\alpha) = \frac{-e^{(3-2\alpha)(t-1)} + 4(3-2\alpha)e^{-(3-2\alpha)t} + e^{-(3-2\alpha)(t+1)}}{4(3-2\alpha)}. \tag{5.27}$$

By using the same arguments and considering (5.19), we will obtain

$$u^{*^r}(t,\alpha) = -\frac{1}{2}e^{\alpha(t-1)}, \tag{5.28}$$

$$x^{*^r}(t,\alpha) = \frac{-e^{\alpha(t-1)} + 4\alpha e^{-\alpha t} + e^{-\alpha(t+1)}}{4\alpha}. \tag{5.29}$$

As we see in Figure 2, $u^{*^l}(t,\alpha)$ is a decreasing function of α, and $u^{*^r}(t,\alpha)$ is an increasing function of α. Therefore, $u^{*^l}(t,\alpha)$ and $u^{*^r}(t,\alpha)$ are not satisfies the conditions **C1-C2** of Lemma 2.1. Then by using the Definition 4.8 (2) of weak solution we define

$$u^{*^l}(t,\alpha) = -\frac{1}{2}e^{\alpha(t-1)}, \tag{5.30}$$

$$u^{*^r}(t,\alpha) = -\frac{1}{2}e^{(3-2\alpha)(t-1)}. \tag{5.31}$$

Also we observe that, $x^{*^l}(t,\alpha)$ is an increasing function of α, and $x^{*^r}(t,\alpha)$ is a decreasing function of α, which mean that the conditions **C1-C2** of Lemma 2.1 are satisfied. Moreover, for all $t \in [0,1]$ the condition **C3** of Lemma 2.1 is satisfied

$$\begin{aligned}
u^{*^l}(t,1) &= u^{*^r}(t,1) = -\frac{1}{2}e^{(t-1)},\\
x^{*^l}(t,1) &= x^{*^r}(t,1) = \frac{-e^{(t-1)} + 4e^{-t} + e^{-(t+1)}}{4}.
\end{aligned}$$

Now the α-level set of optimal fuzzy control $\tilde{u}^*(t)$ and optimal fuzzy state $\tilde{x}^*(t)$ are characterized, respectively, by

$$\begin{aligned}
\tilde{u}^*(t)[\alpha] &= \left[u^{*^r}(t,\alpha), u^{*^l}(t,\alpha)\right] = \left[-\frac{1}{2}e^{\alpha(t-1)}, -\frac{1}{2}e^{(3-2\alpha)(t-1)}\right].\\
\tilde{x}^*(t)[\alpha] &= \left[x^{*^l}(t,\alpha), x^{*^r}(t,\alpha)\right],\\
&= \left[\frac{-e^{(3-2\alpha)(t-1)} + 4(3-2\alpha)e^{-(3-2\alpha)t} + e^{-(3-2\alpha)(t+1)}}{4(3-2\alpha)},\right.\\
&\qquad \left.\frac{-e^{\alpha(t-1)} + 4\alpha e^{-\alpha t} + e^{-\alpha(t+1)}}{4\alpha}\right].
\end{aligned}$$

Therefore, the above solutions are weak solutions of problem (5.16).

Example 5.3. Find the fuzzy control that minimize

$$\begin{aligned}
\tilde{J}(\tilde{u}(t)) &= \tilde{x}(1) \oplus (FR)\int_0^1 \tilde{u}^2(t)dt, \\
\text{subject to:} \quad \dot{\tilde{x}}(t) &= \tilde{u}(t) \oplus (0,1,3)\tilde{x}(t), \quad t \in [0,1], \qquad (5.32)\\
\tilde{x}(0) &= \tilde{1} = (1,1,1), \quad \tilde{x}(t_1) \text{ is free.}
\end{aligned}$$

Solution. First we formulate type-1 fuzzy Hamiltonian function as

$$\tilde{H}(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), t) = \tilde{u}^2(t) \oplus \tilde{\lambda}(t) \odot [\tilde{u}(t) \oplus (0,1,3)\tilde{x}(t)], \tag{5.33}$$

then we have

$$H^l(x^l, u^l, \lambda^l, t, \alpha) = u^{l^2}(t,\alpha) + \lambda^l(t,\alpha)\left[u^l(t,\alpha) + \alpha x^l(t,\alpha)\right], \tag{5.34}$$

$$H^r(x^r, u^r, \lambda^r, t, \alpha) = u^{r^2}(t,\alpha) + \lambda^r(t,\alpha)\left[u^r(t,\alpha) + (3-2\alpha)x^r(t,\alpha)\right], \tag{5.35}$$

and

$$\begin{aligned}
\left[\phi^l(x^l(1),1), \phi^r(x^r(1),1)\right] &= \left[x^l(1), x^r(1)\right], \\
\left[\left.\frac{\partial \phi^l}{\partial x^l}\right|_{t=1}, \left.\frac{\partial \phi^r}{\partial x^r}\right|_{t=1}\right] &= [1,1].
\end{aligned} \tag{5.36}$$

Theorem(4.4) give the necessary conditions for optimality as

$$\dot{x}^l(t,\alpha) = \frac{\partial H^l}{\partial \lambda^l} = u^l(t,\alpha) + \alpha x^l(t,\alpha), \tag{5.37}$$

$$\dot{\lambda}^l(t,\alpha) = -\frac{\partial H^l}{\partial x^l} = -\alpha\lambda^l(t,\alpha), \tag{5.38}$$

$$0 = \frac{\partial H^l}{\partial u^l} = 2u^l(t,\alpha) + \lambda^l(t,\alpha), \tag{5.39}$$

$$\lambda^l(1,\alpha) = \left.\frac{\partial \phi^l}{\partial x^l}\right|_{t=1} \Rightarrow \lambda^l(1,\alpha) = 1. \tag{5.40}$$

We start by solving the DE (5.38) with the condition (5.40), then we obtain

$$\lambda^{*^l}(t,\alpha) = e^{\alpha(1-t)}. \tag{5.41}$$

Substituting (5.41) into (5.39) gives

$$u^{*^l}(t,\alpha) = -\frac{1}{2}e^{\alpha(1-t)}. \tag{5.42}$$

Substituting (5.42) into differential equation (5.37), and then solve it with the initial condition $x^{*^l}(0,\alpha) = 1$, then we obtain

$$x^{*^l}(t,\alpha) = \frac{e^{\alpha(1-t)} + 4\alpha e^{\alpha t} - e^{\alpha(1+t)}}{4\alpha}. \tag{5.43}$$

By using the same arguments and considering (5.35), we obtain

$$u^{*^r}(t,\alpha) = -\frac{1}{2}e^{(3-2\alpha)(1-t)}, \tag{5.44}$$

$$x^{*^r}(t,\alpha) = \frac{e^{(3-2\alpha)(1-t)} + 4(3-2\alpha)e^{(3-2\alpha)t} - e^{(3-2\alpha)(1+t)}}{4(3-2\alpha)}. \tag{5.45}$$

As we see in Figure 3, $u^{*^l}(t,\alpha)$ is a decreasing function of α, and $u^{*^r}(t,\alpha)$ is an increasing function of α. Therefore, $u^{*^l}(t,\alpha)$ and $u^{*^r}(t,\alpha)$ are not satisfies the conditions **C1-C2** of Lemma 2.1. Then by using the Definition 4.8 (2) of weak solution we define

$$u^{*^l}(t,\alpha) = -\frac{1}{2}e^{(3-2\alpha)(1-t)}, \tag{5.46}$$

$$u^{*^r}(t,\alpha) = -\frac{1}{2}e^{\alpha(1-t)}. \tag{5.47}$$

Similarly, we observe that $x^{*^l}(t,\alpha)$ and $x^{*^r}(t,\alpha)$ are an increasing functions of α, it means that the conditions **C1-C2** of Lemma 2.1 are not satisfied. Then by using the Definition 4.8 (2) of weak solution we define

$$x^{*^l}(t,\alpha) = \frac{e^{\alpha(1-t)} + 4\alpha e^{\alpha t} - e^{\alpha(1+t)}}{4\alpha}, \tag{5.48}$$

$$\begin{aligned} x^{*^r}(t,\alpha) &= 2x^{*^l}(t,1) - x^{*^r}(t,\alpha) = \frac{1}{2}e^{(1-t)} + 2e^t - \frac{1}{2}e^{(1+t)} \\ &- \left(\frac{e^{(3-2\alpha)(1-t)} + 4(3-2\alpha)e^{(3-2\alpha)t} - e^{(3-2\alpha)(1+t)}}{4(3-2\alpha)}\right). \end{aligned} \tag{5.49}$$

Moreover, for all $t \in [0,1]$ the condition **C3** of Lemma(2.1) is satisfied

$$\begin{aligned} u^{*^l}(t,1) &= u^{*^r}(t,1) = -\frac{1}{2}e^{(1-t)}, \\ x^{*^l}(t,1) &= x^{*^r}(t,1) = \frac{1}{4}e^{(1-t)} + e^t - \frac{1}{4}e^{(1+t)}. \end{aligned}$$

Now the α-level set of optimal fuzzy control $\tilde{u}^*(t)$ and optimal fuzzy state $\tilde{x}^*(t)$ are characterized, respectively, by

$$\begin{aligned} \tilde{u}^*(t)[\alpha] &= \left[u^{*^r}(t,\alpha), u^{*^l}(t,\alpha)\right] = \left[-\frac{1}{2}e^{(3-2\alpha)(1-t)}, -\frac{1}{2}e^{\alpha(1-t)}\right], \\ \tilde{x}^*(t)[\alpha] &= \left[x^{*^l}(t,\alpha), x^{*^r}(t,\alpha)\right], \\ &= \left[\frac{e^{\alpha(1-t)} + 4\alpha e^{\alpha t} - e^{\alpha(1+t)}}{4\alpha}, \frac{1}{4}e^{(1-t)} + e^t - \frac{1}{4}e^{(1+t)}\right. \\ &- \left.\left(\frac{e^{(3-2\alpha)(1-t)} + 4(3-2\alpha)e^{(3-2\alpha)t} - e^{(3-2\alpha)(1+t)}}{4(3-2\alpha)}\right)\right]. \end{aligned}$$

Therefore, the above solutions are weak solutions of problem (5.32).

6. The Conclusion

In this paper, we considered the fuzzy optimal control problem with state conditions at the final time. In order to propose a solution method for this problem, two types of fuzzy Hamiltonian functions were defined based on Remark 4.2 and Remark 4.5. Then, the necessary conditions for our problem were established in Theorem 4.4 and Theorem 4.7, respectively. Additionally, to assure that the solutions of fuzzy optimal control problems with state conditions at the final time are fuzzy functions, we proposed the concepts of strong and weak solutions of our problem in Definition 4.8. Finally, three examples were used to summarize and highlight the main results of this paper, more specifically, Example (5.1), shows that although Theorem 4.4 provided the necessary conditions based on the restrictions imposed on the fuzzy objective functional and Theorem 4.7 provided the necessary conditions without this restrictions, but, the problem has the same solutions.

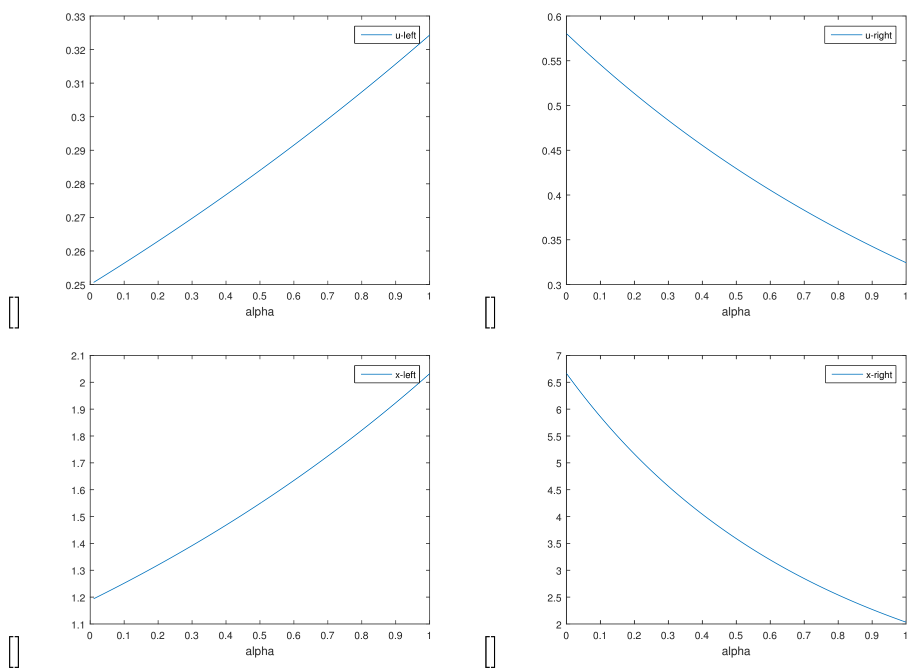

Figure 1. Example 5.1 $(a)u^{*^l}(0.5,\alpha)$ $(b)u^{*^r}(0.5,\alpha)$ $(c)x^{*^l}(0.5,\alpha)$ $(d)x^{*^r}(0.5,\alpha)$.

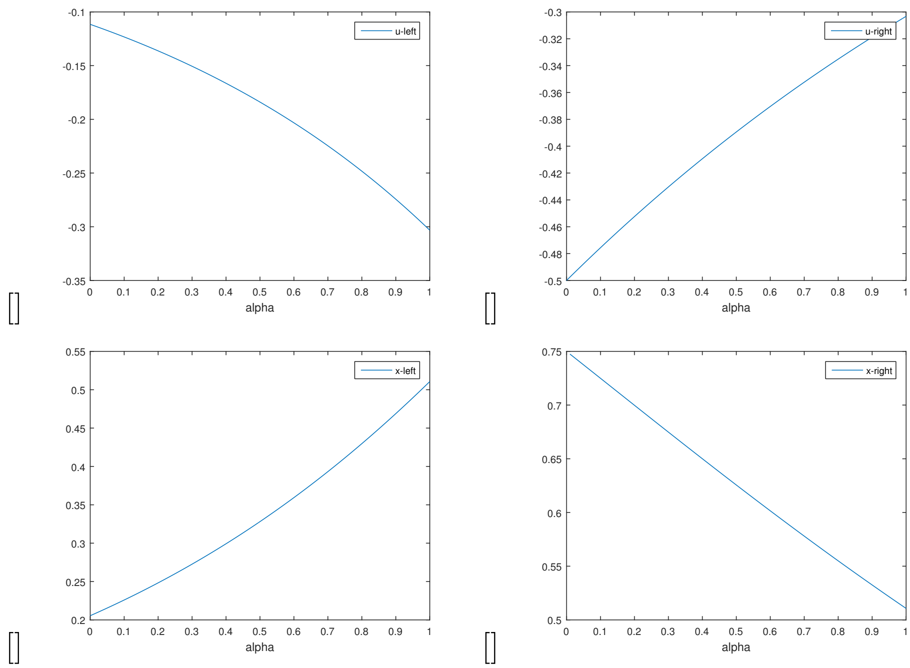

FIGURE 2. Example 5.2 $(a)u^{*^l}(0.5,\alpha)$ $(b)u^{*^r}(0.5,\alpha)$ $(c)x^{*^l}(0.5,\alpha)$ $(d)x^{*^r}(0.5,\alpha)$.

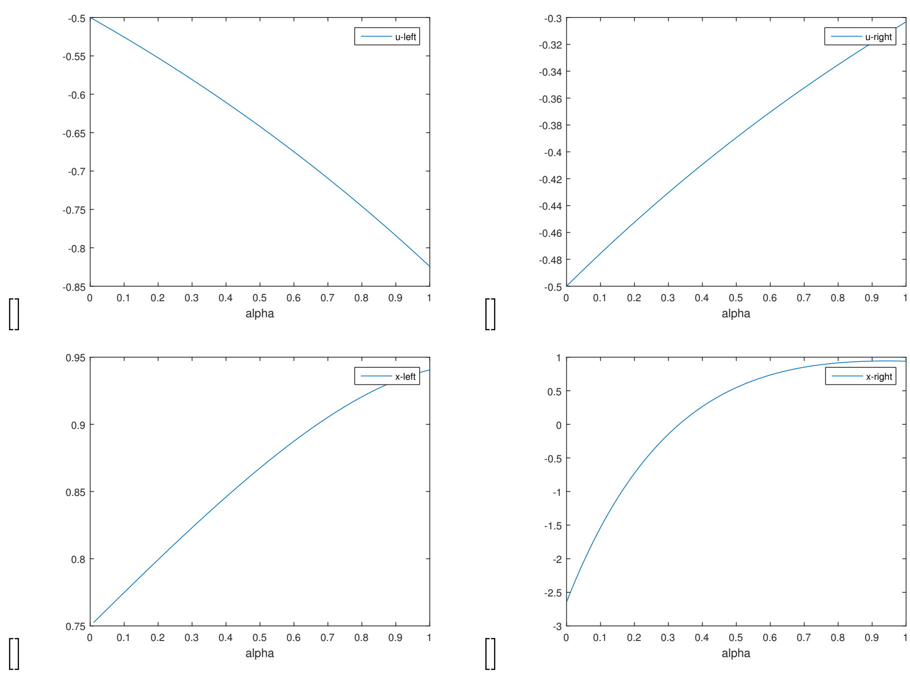

FIGURE 3. Example 5.3 $(a)u^{*^l}(0.5,\alpha)$ $(b)u^{*^r}(0.5,\alpha)$ $(c)x^{*^l}(0.5,\alpha)$ $(d)x^{*^r}(0.5,\alpha)$.

References

[1] S. Lenhart and J.T. Workman, *Optimal control applied to biological models*, Chapman and Hall/CRC, (2007).

[2] C.C. Remsing, *Lecture notes in Linear Control*, Phodes University South Africa(2006).

[3] J.J. Buckley and T. Feuring, *Introduction to fuzzy partial differential equations*, Fuzzy Sets Sys. **105** (1999), 241-248.

[4] S.S.L. Chang and L.A. Zadeh, *On fuzzy mappings and control*, IEEE Trans. SMC. **1** (1972), 30-34.

[5] D. Dubois and H. Prade, *Operation on fuzzy numbers*, Int. J. Systems Sci. **9** (1978), 613-626.

[6] R. Goetschel and W. Voxman, *Elementary fuzzy calculus*, Fuzzy Sets Sys. **18** (1986), 31-43.

[7] B. Bede, S.G. Gal, *Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations*, Fuzzy Sets Sys. **115** (2005), 581-599.

[8] B. Farhadinia, *Pontryagin's minimum principle for fuzzy optimal control problems*, Iranian. J. Fuzzy Sys. **11** (2014), 27-43.

[9] O.S. Fard, J. Soolaki and D.F.M. Torres, *A necessary condition of pontryagin type for fuzzy fractional optimal control problems*, Discrete Contin. Dyn. Syst. Ser. S, in press.

[10] B. Farhadinia, *Necessary optimality conditions for fuzzy variational problems*, Information Science. **181** (2011), 1348-1357.

[11] C.H. Hyun, C.W. Park and S. Kim, *Takagi-Sugeno fuzzy model based indirect adaptive fuzzy observer and controller design*, Information Science. **180** (2010), 2314-2327.

[12] J. Soolaki and O.S. Fard, *A necessary condition of Pontryagin type for fuzzy control systems*, Comput. Appl. Math. **37** (2018), 1263-1278.

[13] T. Takag and M. Sugeno, *Fuzzy identification of systems and its applications to modeling and control*, IEEE Trans. o SMC. **15** (1985), 116-132.

[14] O.S. Fard and M. Ramezanzadeh, *Optimality conditions for optimal control under parametric representation of interval uncertainty*, Int. J. Sys. Control Commun. **9** (2018), 210-229.

[15] F. Dimiter and A. Plamen, *Fuzzy optimal control*, Fuzzy Sets Sys. **47** (1992), 151-156.

[16] P.N Dinh and T.P Van, *The fuzzy dynamic programming problems*, J. Intelligent & Fuzzy Sys. **30** (2016), 1663-1674.

[17] M. Najariyan and M.H. Farahi, *Optimal control of fuzzy linear controled system with fuzzy initial conditions*, Iranian. J. Fuzzy Sys. **10** (2013), 21-35.

[18] A.M. Mustafa, Z.T. Gong and M. Osman, *Fuzzy optimal control problem of several variables*, Adv. Math. Phys. (2019), https://doi.org/10.1155/2019/2182640.

[19] Y.Z. Zhao and Y.G Zhu, *Fuzzy optimal control of linear quadratic models*, Comput. Math. Appl. **60** (2010), 67-73.

[20] L.A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338-353.

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